Empirical Implications of Bequest Motives[*]

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Abstract

We consider four models of familial altruism to account for intergenerational transfers that are necessary to account for the observed wealth distribution and capital accumulation. The first is a pure altruism model in which households directly value the utility of their beneficiaries. The other three are impure altruism models, where only the bequest to the next generation enters the household's utility function. In two of these, households value the expected utility from the bequest—either discounted or not discounted in the same way as consumption. In the third, households value only the utility of a terminal bequest, defined as the bequest left if they reach the maximum possible age. In all four cases, we calibrate the model in general equilibrium to see which model is most consistent with empirical data on capital accumulation and lifecycle patterns of consumption and saving. The pure altruism model requires an implausibly low generational discount factor, close to the interest rate compounded over an average lifespan, to match the data. The three impure altruism models actually result in first-order conditions that are approximately the same over most of the lifecycle. However, the strictly terminal bequest motive is far more tractable than the two expected bequest motive models, which both impose a counterfactual endogenous borrowing constraint. With the exception of cases where the expected-bequest models

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sharply contradict the data, all three impure altruism models produce qualitatively similar predictions, and their quantitative differences are likely too small to serve as a basis for model selection.

Keywords: bequest motives, pure altruism, impure altruism, warm glow, intergenerational transfers, capital accumulation, wealth distribution

1 Introduction

Intergenerational transfers account for a substantial share of the capital stock in the United States, as first demonstrated by Kotlikoff and Summers (1981). Because of this, macroeconomic dynamics can be highly sensitive to the motives behind these transfers-—particularly why some households choose to pass on so much wealth to later generations rather than consuming all of their lifetime income.

There are two dominant models of familial altruism used to model bequests. In the pure altruism approach, households include the utility of their children directly in their own utility function, which means that a household's utility depends on the utility of all their descendants. With impure altruism, households instead get utility from the provision of goods and services to other family members, so utility only depends on what happens during the household's lifetime.

Most empirical investigations into bequests, however, find little statistical evidence to reject the hypothesis that bequests are accidental (see, e.g., Hurd (1987); Kopczuk and Lupton (2007)). A plausible narrative is that many people are intentional about wanting to leave something to their heirs, yet they often describe a lexicographic rule in which Plan A is to first secure their own consumption and healthcare needs and Plan B is to bequeath whatever remains to their children when they die. Such behavior cannot be reconciled with a bequest motive in a standard utility function. Yaari (1965) showed that a rational will for someone with continuous preferences[1] would divide wealth into a portion to fund one's own consumption and a portion destined to be left to heirs. This behavior is, however, consistent with discontinuous preferences, which unfortunately are not compatible with standard solution techniques.

This difficulty perhaps ought not to be so surprising. Empirical puzzles like the equity premium puzzle (Mehra and Prescott (1985)) have already demonstrated the difficulty of representing preferences over consumption with utility functions. The standard utility framework that sums over the lifecycle a function of consumption during each period re-

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[1] Let x_n and y_n be sequences of bundles that converge to x and y respectively. The preference relation $\succsim$ is continuous if $x_n \succsim y_n$ for all n implies that $x \succsim y$.

quires that the elasticity of intertemporal substitution be the inverse of risk aversion. While some, like Epstein and Zin (1989,1991), have played with more complicated formulations that are free of this constraint, most economists, especially macroeconomists, choose simplicity over authenticity and make do by restricting the period utility function to the constant relative risk aversion (CRRA) class.

In this paper we apply the same perspective to bequest motives. We calibrate general equilibrium models with various common formulations of bequest motives to see how compatible they are with stylized facts about capital accumulation, the dynamics of consumption over the lifecycle, and familial behavior.

Of the four models that we consider, one is counterintuitive and probably best understood in terms of bounded rationality: a warm-glow bequest motive in which utility derives solely from the bequest that would be left if the household survives to the terminal period. It is more typical to assume that households value the expected utility that would come from the specific bequest that is left whenever they die. Since this bequest will be a function of remaining wealth, this form of bequest motive is equivalent to adding wealth directly to the utility function as in Carroll's (2000) capitalist spirit model. Carroll added wealth to the utility function to account for why the rich save so much, but doing this in a lifecycle model can induce households to defer consumption to very late in life, which is more consistent with optimal irrational behavior (Feigenbaum, Gahraminov, and Tang (2013)) than what we observe among most households.

Including only the utility from a possible terminal bequest may be aesthetically displeasing, but it solves many practical problems. Most significantly, we can get an analytic solution to the household problem, reducing the problem of finding a general equilibrium to the solution of a single equaiton. This formulation of the bequest motive is also consistent with the fact that most households do not purchase life insurance, a major puzzle for the other models.

Quantitatively, we do not find a substantial difference in behavior between the three impure altruism models. What difference there is can largely be accounted for by the endogenous borrowing constraint in the expected bequest motive models.

With regard to pure altruism models, their deficiency is that, empirically, we do not in fact accumulate much capital. A cohort that cares about the welfare of its progeny at least as much as it cares about its own welfare ought to save far more than we do. If we ignore economic growth, as we do in the present paper, the existing capital-output ratio is only consistent with a generational discount factor roughly equal to the interest rate compounded over an average lifespan. This is less weight than slaves were given by the U.S.

Constitution of 1787.[2]

The weighting of the bequest motive in the impure altruism models is of the same order of magnitude as the generational discount factor in the pure altruism model, but the interpretation is different. Putting a low weight on the utility you get from giving stuff to your kids does not necessarily mean you hate your kids, especially if you are trying to raise them to be independent. On the other hand, putting a low weight on the total utility enjoyed by your kids implies that when you have to make a tradeoff between their welfare and your own welfare then you would generally always favor your welfare. In spite of the name, pure altruism results in very selfish behavior.

The paper is organized as follows. In Section 2 we consider first the mathematically simpler case of impure altruism. Then, applying some of the results from the previous section, we explore the case of pure altruism in Section 3.

2 The Model with Impure Altruism

2.1 The Household Problem

A household that lives $T + 1$ periods maximizes

$$U = \sum_{t=0}^{T} \left[\beta^t Q_t u(c_t) + D_{t+1} u(b_{t+1})\right]$$

subject to

$$c_t + b_{t+1} = y_t + R b_t,$$

where $b_0 = 0$ and $b_{T+1} \geq 0$. The survival probabilities Q_t satisfy $1 \geq Q_0 \geq Q_1 \geq \cdots \geq Q_T > 0$. We assume here a general sequence of weights $D_t \geq 0$ on the instances of wealth in the utility function since these may take various forms depending on the context.

$$\mathcal{L} = \sum_{t=0}^{T} [\beta^t Q_t u(c_t) + D_{t+1} u(b_{t+1}) + \lambda_t (y_t + R b_t - c_t - b_{t+1})].$$

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[2] Accounting for economic growth means that the generational discount factor consistent with macro data depends on the elasticity of intertemporal substitution. If this is low enough, then we can get a generational discount factor of one or more, but then we have a problem with how savings rates respond to interest rates.

The first-order conditions are for $t = 0, \dots, T$,

$$\frac{\partial \mathcal{L}}{\partial c_t} = \beta^t Q_t u'(c_t) - \lambda_t = 0 \tag{1}$$

For $t = 0, \dots, T-1$,

$$\frac{\partial \mathcal{L}}{\partial b_{t+1}} = D_{t+1} u'(b_{t+1}) + R\lambda_{t+1} - \lambda_t = 0. \tag{2}$$

$$\frac{\partial \mathcal{L}}{\partial b_{T+1}} = D_{T+1} u'(b_{T+1}) - \lambda_T = 0. \tag{3}$$

Thus we end up with the generalized Euler equation for $t = 0, \dots, T-1$,

$$\beta^t Q_t u'(c_t) = \beta^{t+1} Q_{t+1} R u'(c_{t+1}) + D_{t+1} u'(b_{t+1})$$

or equivalently

$$u'(c_t) = \beta R \frac{Q_{t+1}}{Q_t} u'(c_{t+1}) + \frac{D_{t+1}}{\beta^t Q_t} u'(b_{t+1}). \tag{4}$$

The terminal Euler equation is

$$u'(c_T) = \frac{D_{T+1}}{\beta^t Q_T} u'(b_{T+1}), \tag{5}$$

which simplifies to

$$c_T = \left(\frac{D_{T+1}}{\beta^T Q_T}\right)^{-\frac{1}{\gamma}} b_{T+1}$$

This generates the sequence of difference equations

$$u'(c_{t+1}) = \frac{u'(c_t) - \frac{D_{t+1}}{\beta^t Q_t} u'(we_t + Rb_t - c_t)}{\beta R \frac{Q_{t+1}}{Q_t}}$$

With CRRA utility, $u(c) = c^{-\gamma}$ for $\gamma > 0$, this becomes

$$c_{t+1} = \left(\beta R \frac{Q_{t+1}}{Q_t}\right)^{1/\gamma} \left[c_t^{-\gamma} - \frac{D_{t+1}}{\beta^t Q_t}(y_t + Rb_t - c_t)^{-\gamma}\right]^{-\frac{1}{\gamma}}. \tag{7}$$

The lifetime budget constraint is

$$\sum_{t=0}^{T} \frac{c_t}{R^t} = \sum_{t=0}^{T} \frac{y_t}{R^t} + \frac{b_{T+1}}{R^T}. \tag{8}$$

$$\frac{\partial c_{t+1}}{\partial c_t} = -\frac{1}{\gamma}\left(\beta R\frac{Q_{t+1}}{Q_t}\right)^{1/\gamma}\left[c_t^{-\gamma} - \frac{D_{t+1}}{\beta^t Q_t}(y_t+Rb_t-c_t)^{-\gamma}\right]^{-\frac{1}{\gamma}-1}\left[-\gamma c_t^{-\gamma-1} - \gamma\frac{D_{t+1}}{\beta^t Q_t}(y_t+Rb_t-c_t)^{-\gamma-1}\right]$$

$$\frac{\partial c_{t+1}}{\partial c_t} = \frac{c_t^{-\gamma-1} + \frac{D_{t+1}}{\beta^t Q_t}(y_t+Rb_t-c_t)^{-\gamma-1}}{c_t^{-\gamma} - \frac{D_{t+1}}{\beta^t Q_t}(y_t+Rb_t-c_t)^{-\gamma}}c_{t+1} > 0$$

$$\frac{\partial c_{t+1}}{\partial D_{t+1}} = -\frac{1}{\gamma}\left(\beta R\frac{Q_{t+1}}{Q_t}\right)^{1/\gamma}\left[c_t^{-\gamma} - \frac{D_{t+1}}{\beta^t Q_t}(y_t+Rb_t-c_t)^{-\gamma}\right]^{-\frac{1}{\gamma}-1}\left(-\frac{1}{\beta^t Q_t}(y_t+Rb_t-c_t)^{-\gamma}\right)$$

$$\frac{\partial c_{t+1}}{\partial D_{t+1}} = \frac{1}{\gamma}\frac{\frac{1}{\beta^t Q_t}(y_t+Rb_t-c_t)^{-\gamma}}{c_t^{-\gamma} - \frac{D_{t+1}}{\beta^t Q_t}(y_t+Rb_t-c_t)^{-\gamma}}c_{t+1} > 0.$$

Suppose $\gamma = 1$. Then we have

$$c_{t+1} = \frac{\beta R\frac{Q_{t+1}}{Q_t}}{\frac{1}{c_t} - \frac{D_{t+1}}{\beta^t Q_t}\frac{1}{y_t+Rb_t-c_t}}.$$

$$c_{t+1} = \frac{\beta R\frac{Q_{t+1}}{Q_t}c_t(y_t+Rb_t-c_t)}{y_t+Rb_t-c_t-\frac{D_{t+1}}{\beta^t Q_t}c_t}$$

$$c_{t+1} = \frac{y_t+Rb_t-c_t}{y_t+Rb_t-\left(1+\frac{D_{t+1}}{\beta^t Q_t}\right)c_t}\beta R\frac{Q_{t+1}}{Q_t}c_t > \beta R c_t$$

Note that if $b_1 < 0$ with $D_1 = 0$ then putting wealth in the utility function will cause a huge change in behavior. We have to choose c_0 so that the b_t are all positive. Thus putting wealth in the utility function imposes an endogenous no-borrowing constraint that is counterfactual.

If the D_t are all positive, we can compute a solution to the household problem by guessing at c_0 and then solving (7) to determine the lifecycle profiles of b_{t+1} and c_t. From Appendix A, we know that there is a threshold $\overline{c}_0 > 0$ such that $c_t, b_{t+1} > 0$ for all $t = 1, \ldots, T$ for $c_0 < \overline{c}_0 > 0$. Moreover, there is a unique $c_0 \in (0, \overline{c}_0)$ such that (5) is satisfied. In practice, the solution for c_0 is usually very close to $\overline{c}_0$, so we have to use bisection methods to find c_0.

2.2 General Equilibrium

Let us suppose that

$$y_t = we_t + B,$$

where w is the wage per efficiency unit, e_t is the productivity of a household of age t, and B is the uniform bequest. For simplicity, since the bequest motive only depends on the leaving of the bequest and not on what the heir does with the bequest, we will assume here that these bequests are distributed uniformly over the whole surviving population.[3]

The labor supply is

$$N = \sum_{t=0}^{T} Q_t e_t.$$

$$K = \sum_{t=0}^{T} Q_t b_{t+1}.$$

Then there is a constant returns to scale production function

$$F(K, N)$$

such that the factor prices are

$$\begin{aligned} w &= F_N(K, N) \\ R &= F_K(K, N) + 1 - \delta. \end{aligned}$$

The bequest balance equation is then

$$B \sum_{t=0}^{T} Q_t = R \sum_{t=0}^{T} (Q_t - Q_{t+1}) b_{t+1},$$

where the left-hand side is the uniform bequest times the total population, and the right-hand side is the total financial wealth left by those who die each period.

2.3 Comparison of Different Parameterizations

We will consider three different parameterizations of the weights D_t on the bequest motive: (i) a discounted expected bequest motive in which

$$D_{t+1} = \rho \beta^{t+1} (Q_t - Q_{t+1}), \tag{9}$$

(ii) an undiscounted expected bequest motive in which

$$D_{t+1} = \rho (Q_t - Q_{t+1}), \tag{10}$$

and (iii) a strictly terminal bequest motive in which $D_{T+1} = \rho$ and $D_{t+1} = 0$ for $t < T$. In all three cases, the D_t will be proportional to $\rho \geq 0$, which quantifies the strength of the

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[3] In Feigenbaum et al. (2024a) and Feigenbaum et al. (2024b), we explore the consequences of the warm glow bequest motive depending on who gets the bequest.

bequest motive relative to the utility from consumption. The first case corresponds to the most common version of a warm glow bequest motive in the literature. The second case accounts for the fact that it is not really clear why we ought to discount utility from the bequest motive since, introspectively, the utility from the bequest ought not to depend on when you die. Finally, the third case reflects the bounded rationality that we may imagine that any bequest we leave will be in the far future.

Note that all three cases have the same Euler equation (4), which can be rewritten

$$c_t^{-\gamma} = \beta R \frac{Q_{t+1}}{Q_t} c_{t+1}^{-\gamma} + \frac{D_{t+1}}{\beta^t Q_t} b_{t+1}^{-\gamma}. \tag{11}$$

For the strictly terminal bequest motive, (11) simplifies to the standard Euler equation with uninsured mortality risk

$$c_t^{-\gamma} = \beta R \frac{Q_{t+1}}{Q_t} c_{t+1}^{-\gamma} \tag{12}$$

for $t < T$. In contrast, the two expected bequest motive models will have the marginal utility from the bequest in (11) for all t. But this distinction is not as significant as it may appear because households will quickly accumulate a large holding of savings, in which case the marginal utility of the bequest will become negligible. Thus, for most of the lifecycle, (12) will be a reasonable approximation to (11). So we should not expect the lifecycle profiles produced by the three impure altruism models to be substantially different except when households are young and savings are low and possibly negative in the case of the strictly terminal bequest motive. Households will endogenously not borrow under (11) if the $D_{t+1} > 0$ for all $t = 1, \ldots, T$.

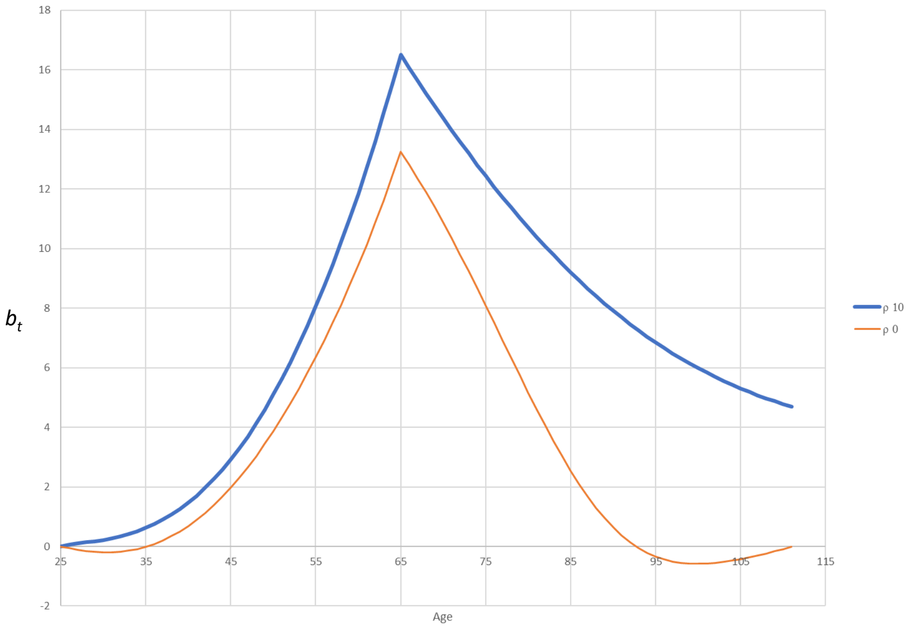
In the following, we calibrate the model with $\alpha = \frac{1}{3}$, $\beta = 0.94$, $\gamma = 1.0$, and $\delta = 0.04$. The discount factor was chosen to get a capital-output ratio of roughly 3.0 years in the absence of any bequest motive. The labor productivity e_t and survival probability Q_t profiles are taken from Feigenbaum (2008).

2.3.1 Discounted Expected Bequest Motive

First, we consider what happens in the discounted expected bequest motive specification. We need a large value of ρ greater than one in order to get intergenerational transfers to contribute substantially to the bequest motive as found by Kotlikoff and Summers (1981). In Figs. 1 and 2, we show the lifecycle savings and consumption profiles with a bequest motive ($\rho = 10$) and without ($\rho = 0$).

Fig. 1 shows that the bequest motive does increase household savings, especially after retirement. As a result, the capital stock increases by 50%, so the bequest motive accounts

Figure 1: Lifecycle Savings Profile With and Without Bequest Motive



Note: This figure compares the lifecycle profiles of savings for the calibration with $\alpha = \frac{1}{3}$, $\beta = 0.94$, $\gamma = 1.0$, $\delta = 0.04$. The profile with the bequest motive has $\rho = 10$.

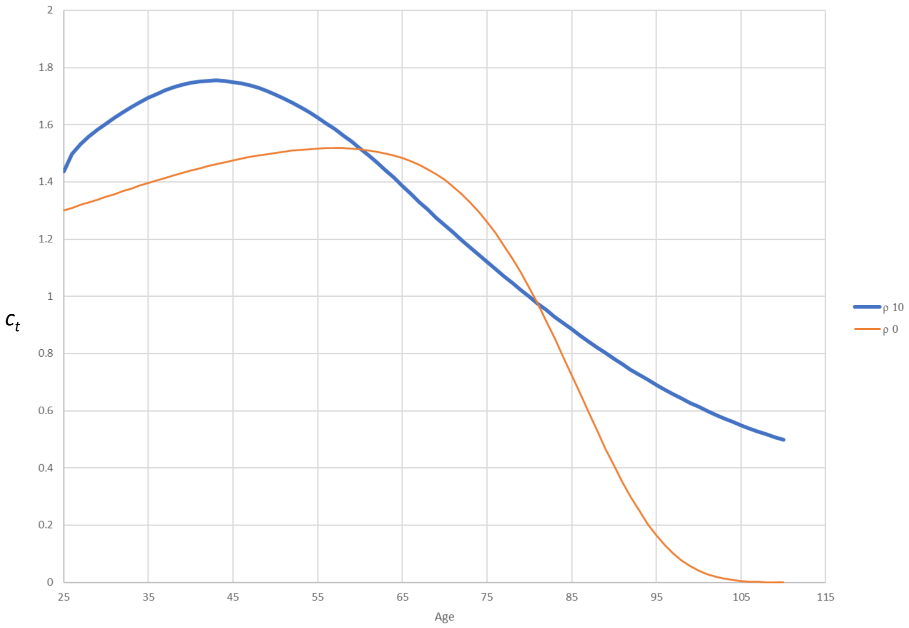
for a third of the capital stock with $\rho = 10$.

While Feigenbaum (2008) previously considered mortality risk with accidental bequests as a cause of the consumption hump, Fig. 2 indicates that a bequest motive can substantially modify the peak age and size of a mortality-induced consumption hump.

2.3.2 Undiscounted Expected Bequest Motive

For most plausible calibrations, the undiscounted expected bequest motive model yields lifecycle profiles qualitative similar to the discounted expected bequest motive model. However, for our chosen calibration, computing the solution to the household equilibrium is extremely difficult because c_0 is indistinguishable from $\overline{c}_0$ at the precisions that are available on most computational platforms. Even with a negligible $\rho > 0$, the endogenous borrowing constraint causes households to accumulate considerable wealth very quickly so that the marginal utility from the bequest motive effectively vanishes, causing the household to conserve this wealth until the maximum possible age, resulting in a spike either in consumption or bequeathed wealth for households at this improbable age. The computational difficulty is determining which use the household allocates this wealth towards. While this is of little import to the household, it matters significantly for determining the general equilibrium.

Figure 2: Lifecycle Consumption Profile With and Without Bequest Motive



Note: This figure compares the lifecycle profiles of consumption for the calibration with $\alpha = \frac{1}{3}$, $\beta = 0.94$, $\gamma = 1.0$, and $\delta = 0.04$. The profile with the bequest motive has $\rho = 10$.

2.3.3 Strictly Terminal Bequest Motive

An advantage of this specification is that we can obtain an analytic expression for all the endogenous quantities except the equilibrium capital stock.

Let us now suppose that $D_{T+1} = \rho$ and $D_t = 0$ for $t < T + 1$. Thus we have

$$U = \sum_{t=0}^{T} \beta^t Q_t u(c_t) + \rho u(b_{T+1})$$

subject to

$$c_t + b_{t+1} = y_t + Rb_t,$$

where $b_0 = 0$ and $b_{T+1} \geq 0$.

$$\sum_{t=0}^{T} \frac{c_t + b_{t+1}}{R^t} = \sum_{t=0}^{T} \frac{y_t + Rb_t}{R^t}$$

$$\sum_{t=0}^{T} \frac{c_t}{R^t} + \sum_{t=0}^{T} \frac{b_{t+1}}{R^t} - \sum_{t=0}^{T} \frac{b_t}{R^{t-1}} = \sum_{t=0}^{T} \frac{y_t}{R^t}$$

$$\sum_{t=0}^{T} \frac{c_t}{R^t} + \sum_{t=0}^{T} \frac{b_{t+1}}{R^t} - \sum_{t=1}^{T} \frac{b_t}{R^{t-1}} = \sum_{t=0}^{T} \frac{y_t}{R^t}$$

$$\sum_{t=0}^{T} \frac{c_t}{R^t} + \sum_{t=0}^{T} \frac{b_{t+1}}{R^t} - \sum_{t=0}^{T-1} \frac{b_{t+1}}{R^t} = \sum_{t=0}^{T} \frac{y_t}{R^t}$$

$$\sum_{t=0}^{T} \frac{c_t}{R^t} + \frac{b_{T+1}}{R^T} = \sum_{t=0}^{T} \frac{y_t}{R^t}.$$

The Lagrangian for the household problem is then

$$\mathcal{L} = \sum_{t=0}^{T} \beta^t Q_t u(c_t) + \rho u(b_{T+1}) + \lambda \left[\sum_{t=0}^{T} \frac{y_t - c_t}{R^t} - \frac{b_{T+1}}{R^T} \right]$$

$$\frac{\partial \mathcal{L}}{\partial c_t} = \beta^t Q_t u'(c_t) - \frac{\lambda}{R^t} = 0$$

$$\frac{\partial \mathcal{L}}{\partial b_{T+1}} = \rho u'(b_{T+1}) - \frac{\lambda}{R^T} = 0$$

$$\lambda = \beta^t R^t Q_t c_t^{-\gamma} = \rho R^T b_{T+1}^{-\gamma}.$$

$$\lambda = Q_0 c_0^{-\gamma}$$

$$Q_0 c_0^{-\gamma} = \beta^t R^t Q_t c_t^{-\gamma}$$

$$c_t^{\gamma} = \frac{\beta^t R^t Q_t}{Q_0} c_0^{\gamma}$$

$$c_t = \left(\frac{\beta^t R^t Q_t}{Q_0} \right)^{\frac{1}{\gamma}} c_0$$

$$Q_0 c_0^{-\gamma} = \rho R^T b_{T+1}^{-\gamma}$$

$$b_{T+1}^{\gamma} = \frac{\rho R^T}{Q_0} c_0^{\gamma}$$

$$b_{T+1} = \left(\frac{\rho R^T}{Q_0} \right)^{\frac{1}{\gamma}} c_0$$

Let

$$H_t = \sum_{s=t}^{T} \frac{y_s}{R^{s-t}},$$

so

$$H_0 = \sum_{t=0}^{T} \frac{y_t}{R^t}.$$

Then we have

$$\sum_{t=0}^{T}\left(\frac{\beta^t R^t Q_t}{Q_0}\right)^{\frac{1}{\gamma}}\frac{c_0}{R^t}+\left(\frac{\rho R^T}{Q_0}\right)^{\frac{1}{\gamma}}\frac{c_0}{R^T}=H_0$$

Thus

$$c_0=\frac{H_0}{\sum_{t=0}^{T}\left(\frac{Q_t}{Q_0}\right)^{1/\gamma}(\beta R^{1-\gamma})^{\frac{t}{\gamma}}+\left(\frac{\rho R^{(1-\gamma)T}}{Q_0}\right)^{\frac{1}{\gamma}}}$$

$$c_t=\frac{\left(\frac{\beta^t R^t Q_t}{Q_0}\right)^{\frac{1}{\gamma}}}{\sum_{s=0}^{T}\left(\frac{Q_s}{Q_0}\right)^{1/\gamma}(\beta R^{1-\gamma})^{\frac{s}{\gamma}}+\left(\frac{\rho R^{(1-\gamma)T}}{Q_0}\right)^{\frac{1}{\gamma}}}H_0$$

$$b_{T+1}=\frac{\left(\frac{\rho R^T}{Q_0}\right)^{\frac{1}{\gamma}}}{\sum_{t=0}^{T}\left(\frac{Q_t}{Q_0}\right)^{1/\gamma}(\beta R^{1-\gamma})^{\frac{t}{\gamma}}+\left(\frac{\rho R^{(1-\gamma)T}}{Q_0}\right)^{\frac{1}{\gamma}}}H_0 \tag{13}$$

$$\frac{b_{T+1}}{c_T}=\frac{\left(\frac{\rho R^T}{Q_0}\right)^{\frac{1}{\gamma}}}{\left(\frac{\beta^T R^T Q_T}{Q_0}\right)^{\frac{1}{\gamma}}}=\left(\frac{\rho}{\beta^T Q_T}\right)^{\frac{1}{\gamma}}$$

The budget constraint is

$$b_{t+1}=y_t-c_t+Rb_t$$

$$\sum_{s=0}^{t}\frac{b_{s+1}}{R^s}=\sum_{s=0}^{t}\frac{y_s-c_s}{R^s}+R\sum_{s=0}^{t}\frac{b_s}{R^s}$$

$$\sum_{s=0}^{t}\frac{b_{s+1}}{R^s}-\sum_{s=0}^{t}\frac{b_s}{R^{s-1}}=\sum_{s=0}^{t}\frac{y_s-c_s}{R^s}$$

$$\sum_{s=0}^{t}\frac{b_{s+1}}{R^s}-\sum_{s=1}^{t}\frac{b_s}{R^{s-1}}=\sum_{s=0}^{t}\frac{y_s-c_s}{R^s}$$

$$\sum_{s=0}^{t}\frac{b_{s+1}}{R^s}-\sum_{s=0}^{t-1}\frac{b_{s+1}}{R^s}=\sum_{s=0}^{t}\frac{y_s-c_s}{R^s}$$

$$\frac{b_{t+1}}{R^t}=\sum_{s=0}^{t}\frac{y_s-c_s}{R^s}$$

$$b_{t+1}=\sum_{s=0}^{t}\frac{y_s-c_s}{R^{s-t}}$$

$$b_{T+1} = \sum_{t=0}^{T} \frac{y_t - c_t}{R^{t-T}} = R^T \sum_{t=0}^{T} \frac{y_t}{R^t} - R^T \sum_{t=0}^{T} \frac{1}{R^t} \frac{\left(\frac{\beta^t R^t Q_t}{Q_0}\right)^{\frac{1}{\gamma}}}{\sum_{s=0}^{T} \left(\frac{Q_s}{Q_0}\right)^{1/\gamma} (\beta R^{1-\gamma})^{\frac{s}{\gamma}} + \left(\frac{\rho R^{(1-\gamma)T}}{Q_0}\right)^{\frac{1}{\gamma}}} H_0$$

$$b_{T+1} = R^T H_0 - R^T \sum_{t=0}^{T} \frac{\left(\frac{Q_t}{Q_0}\right)^{\frac{1}{\gamma}} (\beta R^{1-\gamma})^{\frac{t}{\gamma}}}{\sum_{s=0}^{T} \left(\frac{Q_s}{Q_0}\right)^{1/\gamma} (\beta R^{1-\gamma})^{\frac{s}{\gamma}} + \left(\frac{\rho R^{(1-\gamma)T}}{Q_0}\right)^{\frac{1}{\gamma}}} H_0$$

$$b_{T+1} = \left[1 - \frac{\sum_{t=0}^{T} \left(\frac{Q_t}{Q_0}\right)^{\frac{1}{\gamma}} (\beta R^{1-\gamma})^{\frac{t}{\gamma}}}{\sum_{s=0}^{T} \left(\frac{Q_s}{Q_0}\right)^{1/\gamma} (\beta R^{1-\gamma})^{\frac{s}{\gamma}} + \left(\frac{\rho R^{(1-\gamma)T}}{Q_0}\right)^{\frac{1}{\gamma}}}\right] R^T H_0$$

$$b_{T+1} = \frac{\left(\frac{\rho R^{(1-\gamma)T}}{Q_0}\right)^{\frac{1}{\gamma}}}{\sum_{s=0}^{T} \left(\frac{Q_s}{Q_0}\right)^{1/\gamma} (\beta R^{1-\gamma})^{\frac{s}{\gamma}} + \left(\frac{\rho R^{(1-\gamma)T}}{Q_0}\right)^{\frac{1}{\gamma}}} R^T H_0$$

$$b_{T+1} = \frac{\left(\frac{\rho R^{T}}{Q_0}\right)^{\frac{1}{\gamma}}}{\sum_{s=0}^{T} \left(\frac{Q_s}{Q_0}\right)^{1/\gamma} (\beta R^{1-\gamma})^{\frac{s}{\gamma}} + \left(\frac{\rho R^{(1-\gamma)T}}{Q_0}\right)^{\frac{1}{\gamma}}} H_0,$$

which matches (13).

$$b_{t+1} = \sum_{z=0}^{t} \frac{y_z - c_z}{R^{z-t}}$$

$$c_t = \frac{\left(\frac{\beta^t R^t Q_t}{Q_0}\right)^{\frac{1}{\gamma}}}{\sum_{s=0}^{T} \left(\frac{Q_s}{Q_0}\right)^{1/\gamma} (\beta R^{1-\gamma})^{\frac{s}{\gamma}} + \left(\frac{\rho R^{(1-\gamma)T}}{Q_0}\right)^{\frac{1}{\gamma}}} H_0$$

$$b_{t+1} = \sum_{z=0}^{t} \frac{1}{R^{z-t}} \left[y_z - \frac{\left(\frac{\beta^z R^z Q_z}{Q_0}\right)^{\frac{1}{\gamma}}}{\sum_{s=0}^{T} \left(\frac{Q_s}{Q_0}\right)^{1/\gamma} (\beta R^{1-\gamma})^{\frac{s}{\gamma}} + \left(\frac{\rho R^{(1-\gamma)T}}{Q_0}\right)^{\frac{1}{\gamma}}} H_0\right]$$

$$b_{t+1} = R^t \left[\sum_{z=0}^{t} \frac{y_z}{R^z} - H_0 \frac{\sum_{z=0}^{t} \left(\frac{Q_z}{Q_0}\right)^{1/\gamma} (\beta R^{1-\gamma})^{\frac{z}{\gamma}}}{\sum_{s=0}^{T} \left(\frac{Q_s}{Q_0}\right)^{1/\gamma} (\beta R^{1-\gamma})^{\frac{s}{\gamma}} + \left(\frac{\rho R^{(1-\gamma)T}}{Q_0}\right)^{\frac{1}{\gamma}}}\right]$$

$$b_{t+1} = R^t \left[\frac{\sum_{z=t+1}^{T} \left(\frac{Q_z}{Q_0}\right)^{1/\gamma} (\beta R^{1-\gamma})^{\frac{z}{\gamma}} + \left(\frac{\rho R^{(1-\gamma)T}}{Q_0}\right)^{\frac{1}{\gamma}}}{\sum_{s=0}^{T} \left(\frac{Q_s}{Q_0}\right)^{1/\gamma} (\beta R^{1-\gamma})^{\frac{s}{\gamma}} + \left(\frac{\rho R^{(1-\gamma)T}}{Q_0}\right)^{\frac{1}{\gamma}}} \sum_{i=0}^{t} \frac{y_i}{R^i} \right.$$
$$\left. - \frac{\sum_{z=0}^{t} \left(\frac{Q_z}{Q_0}\right)^{1/\gamma} (\beta R^{1-\gamma})^{\frac{z}{\gamma}}}{\sum_{s=0}^{T} \left(\frac{Q_s}{Q_0}\right)^{1/\gamma} (\beta R^{1-\gamma})^{\frac{s}{\gamma}} + \left(\frac{\rho R^{(1-\gamma)T}}{Q_0}\right)^{\frac{1}{\gamma}}} \sum_{i=t+1}^{T} \frac{y_i}{R^i} \right]$$

Let us define for $t = 0, ..., T$,

$$m_t = \left(\frac{Q_t}{Q_0}\right)^{1/\gamma} (\beta R)^{\frac{t}{\gamma}} \tag{14}$$

and

$$m_{T+1} = \left(\frac{\rho R^T}{Q_0}\right)^{\frac{1}{\gamma}}. \tag{15}$$

For $t = 0, ..., T$,

$$M_t = \sum_{i=0}^{t} \frac{m_t}{R^t} \tag{16}$$

and

$$M_{T+1} = M_T + \frac{m_{T+1}}{R^T}.$$

Thus

$$c_t = \frac{m_t}{M_{T+1}} H_0$$
$$b_{T+1} = \frac{m_{T+1}}{M_{T+1}} H_0.$$

$$b_{t+1} = \frac{R^t}{M_{T+1}} \left[(M_{T+1} - M_t) \sum_{i=0}^{t} \frac{we_i + B}{R^i} - M_t \sum_{i=t+1}^{T} \frac{we_i + B}{R^i} \right].$$

$$b_{t+1} = R^t \left[\sum_{i=0}^{t} \frac{we_i + B}{R^i} - \frac{M_t}{M_{T+1}} \sum_{i=0}^{T} \frac{we_i + B}{R^i} \right]. \tag{17}$$

One advantage to the strictly terminal bequest model is that we can analytically decompose what the capital stock would be without a bequest motive. We can set B and ρ to 0, to obtain

$$b^0_{t+1} = wR^t \left[\sum_{i=0}^{t} \frac{e_i}{R^i} - \frac{M_t}{M_T} \sum_{i=0}^{T} \frac{e_i}{R^i} \right].$$

Note that

$$b^0_{T+1} = wR^T \left[\sum_{i=0}^{T} \frac{e_i}{R^i} - \frac{M_T}{M_T} \sum_{i=0}^{T} \frac{e_i}{R^i} \right] = 0,$$

as it should be. We can also rewrite (17) as

$$\begin{aligned} b_{t+1} &= wR^t \left[\sum_{i=0}^{t} \frac{e_i}{R^i} - \frac{M_t}{M_{T+1}} \sum_{i=0}^{T} \frac{e_i}{R^i} \right] + BR^t \left[\sum_{i=0}^{t} \frac{1}{R^i} - \frac{M_t}{M_{T+1}} \sum_{i=0}^{T} \frac{1}{R^i} \right] \\ &= wR^t \left[\sum_{i=0}^{t} \frac{e_i}{R^i} - \frac{M_t}{M_{T+1}} \sum_{i=0}^{T} \frac{e_i}{R^i} \right] + BR^t \left[\frac{1 - R^{-t-1}}{1 - \frac{1}{R}} - \frac{M_t}{M_{T+1}} \frac{1 - R^{-T-1}}{1 - \frac{1}{R}} \right] \end{aligned}$$

$$b_{t+1} = wR^t \left[\sum_{i=0}^{t} \frac{e_i}{R^i} - \frac{M_t}{M_{T+1}} \sum_{i=0}^{T} \frac{e_i}{R^i} \right] + \frac{BR^{t+1}}{R-1} \left[1 - R^{-t-1} - \frac{M_t}{M_{T+1}} (1 - R^{-T-1}) \right]$$

$$b_t = wR^{t-1} \left[\sum_{i=0}^{t-1} \frac{e_i}{R^i} - \frac{M_{t-1}}{M_{T+1}} \sum_{i=0}^{T} \frac{e_i}{R^i} \right] + \frac{BR^t}{R-1} \left[1 - R^{-t} - \frac{M_{t-1}}{M_{T+1}} (1 - R^{-T-1}) \right]$$

The bequest balance equation is

$$B \sum_{t=0}^{T} Q_t = R \sum_{t=0}^{T} (Q_t - Q_{t+1}) b_{t+1}.$$

Let

$$P = \sum_{t=0}^{T} Q_t. \tag{18}$$

Let us define

$$b^e_t = wR^{t-1} \left[\sum_{i=0}^{t-1} \frac{e_i}{R^i} - \frac{M_{t-1}}{M_{T+1}} \sum_{i=0}^{T} \frac{e_i}{R^i} \right]$$

$$z_t = \frac{BR^t}{R-1} \left[1 - R^{-t} - \frac{M_{t-1}}{M_{T+1}} (1 - R^{-T-1}) \right]$$

$$PB = R \sum_{t=0}^{T} (Q_t - Q_{t+1})(b^e_{t+1} + z_{t+1} B)$$

$$B = \frac{R \sum_{t=0}^{T} (Q_t - Q_{t+1}) b^e_{t+1}}{P - R \sum_{t=0}^{T} (Q_t - Q_{t+1}) z_{t+1}}$$

3 The Model with Pure Altruism

Let us suppose we have a household with pure altruism that solves the Bellman equation

$$v(b_0) = \max_{\{c_t, b_{t+1}\}_{t=0}^T} \sum_{t=0}^{T} [Q_t \beta^t u(c_t) + \rho(Q_t - Q_{t+1}) v(b_{t+1})]$$

subject to

$$c_t + b_{t+1} = w e_t + R b_t,$$

where $Q_0 = 1$ and $Q_{T+1} = 0$. The Lagrangian is

$$\mathcal{L} = \sum_{t=0}^{T} \left\{ [Q_t \beta^t u(c_t) + \rho(Q_t - Q_{t+1}) v(b_{t+1})] + \lambda_t [y_t + R b_t - c_t - b_{t+1}] \right\}.$$

The first-order conditions are for $t = 0, ..., T$,

$$\frac{\partial \mathcal{L}}{\partial c_t} = \beta^t Q_t u'(c_t) - \lambda_t = 0.$$

For $t = 0, ..., T-1$,

$$\frac{\partial \mathcal{L}}{\partial b_{t+1}} = \rho(Q_t - Q_{t+1}) v'(b_{t+1}) + \lambda_{t+1} R - \lambda_t = 0.$$

We must treat the terminal saving separately:

$$\frac{\partial \mathcal{L}}{\partial b_{T+1}} = \rho Q_T v'(b_{T+1}) - \lambda_T = 0.$$

By the Envelope Theorem,

$$v'(b_0) = \frac{\partial \mathcal{L}}{\partial b_0} = \lambda_0 R = R u'(c_0).$$

Because of the clear parent-child relationship, we cannot avoid dealing with the uncertainty like we can the impure altruism model if there is mortality risk. But suppose $Q_T = 1$. Then there is no uncertainty. We know that c_0 will be the same for everyone in the steady state, so $v'(b_{T+1}) = R u'(c_0)$.

$$\rho R u'(c_0) = \lambda_T$$

$$\lambda_{t+1} = \frac{\lambda_t}{R}$$

so

$$\lambda_T = \frac{\lambda_0}{R^T}$$

$$\rho R u'(c_0) = \frac{\lambda_0}{R^T}$$

$$\lambda_0 = u'(c_0),$$

so we have $\rho R^{T+1} = 1$, which is the usual result.

In this model when a household dies, it is replaced by a household of age 0 that starts out with bequest b_0 that is randomly drawn from a distribution $g_0(b_0)$. Let $g_t(b_t)$ be the distribution of b_t for a household of age t. The capital stock will be

$$K = \sum_{t=0}^{T} Q_t \int g_t(b_t) db_t. \tag{19}$$

$$N = \sum_{t=0}^{T} Q_t e_t. \tag{20}$$

Let us suppose more generally that $u(c) = \ln c$ and let us guess that

$$v(b_0) = D \ln(b_0 + H) + G.$$

The Lagrangian is then

$$\mathcal{L} = \sum_{t=0}^{T} \left\{ [Q_t \beta^t \ln(c_t) + \rho(Q_t - Q_{t+1})[D \ln(b_{t+1} + H) + G]] + \lambda_t [y_t + R b_t - c_t - b_{t+1}] \right\}. \tag{21}$$

The first-order conditions are

$$\frac{\partial \mathcal{L}}{\partial c_t} = \frac{Q_t \beta^t}{c_t} - \lambda_t = 0$$

For $t = 0, ..., T-1$,

$$\frac{\partial \mathcal{L}}{\partial b_{t+1}} = \rho(Q_t - Q_{t+1}) \frac{D}{b_{t+1} + H} - \lambda_t + R\lambda_{t+1} = 0$$

$$\frac{\partial \mathcal{L}}{\partial b_{T+1}} = \rho Q_T \frac{D}{b_{T+1} + H} - \lambda_T = 0$$

$$\rho \frac{Q_T D}{b_{T+1} + H} = \frac{Q_T \beta^T}{c_T}$$

$$b_{T+1} + H = \frac{\rho D}{\beta^T} c_T$$

$$c_T + b_{T+1} = y_T + R b_T$$

$$c_T + b_{T+1} + H = y_T + R b_T + H$$

$$\left(1+\frac{\rho D}{\beta^T}\right)c_T = y_T + Rb_T + H$$

$$c_T = \frac{y_T + Rb_T + H}{1+\frac{\rho D}{\beta^T}} = \frac{\beta^T}{\beta^T+\rho D}(y_T + Rb_T + H)$$

$$b_{T+1} + H = \frac{\rho D}{\beta^T+\rho D}(y_T + Rb_T + H)$$

$$b_{T+1} = \frac{\rho D}{\beta^T+\rho D}(y_T + Rb_T) - \frac{\beta^T}{\beta^T+\rho D}H$$

$$\rho(Q_t - Q_{t+1})\frac{D}{b_{t+1}+H} + R\lambda_{t+1} = \lambda_t$$

$$\rho(Q_t - Q_{t+1})\frac{D}{b_{t+1}+H} + R\frac{Q_{t+1}\beta^{t+1}}{c_{t+1}} = \frac{Q_t\beta^t}{c_t}$$

$$R\frac{Q_{t+1}\beta^{t+1}}{c_{t+1}} = \frac{Q_t\beta^t}{c_t} - \rho(Q_t - Q_{t+1})\frac{D}{y_t + Rb_t - c_t + H}$$

I think we are only going to get an analytic solution if the $y_t = 0$, in which case the consumptions and savings will all be proportional to b_0. In that case, $H = 0$, so the Euler equation is

$$R\frac{Q_{t+1}\beta^{t+1}}{c_{t+1}} = \frac{Q_t\beta^t}{c_t} - \rho(Q_t - Q_{t+1})\frac{D}{Rb_t - c_t}$$

$$c_{t+1} = \frac{Q_{t+1}\beta^{t+1}R}{\frac{Q_t\beta^t}{c_t} - \rho(Q_t - Q_{t+1})\frac{D}{Rb_t - c_t}}$$

$$c_{t+1} = \frac{\frac{Q_{t+1}}{Q_t}\beta R c_t}{1 - \frac{\rho}{\beta^t}\left(1 - \frac{Q_{t+1}}{Q_t}\right)\frac{Dc_t}{Rb_t - c_t}}.$$

Let us guess that

$$c_t = m_t b_0$$

and

$$b_{t+1} = \kappa_{t+1} b_0$$

for $\{m_t\}_{t=0}^T, \{\kappa_{t+1}\}_{t=0}^T \geq 0$. Then we have the system of equations

$$m_{t+1} = \frac{\frac{Q_{t+1}}{Q_t}\beta R m_t}{1 - \frac{\rho}{\beta^t}\left(1 - \frac{Q_{t+1}}{Q_t}\right)\frac{Dm_t}{R\kappa_{t-1} - m_t}}$$

for $t = 0, ..., T-1$.

For $t = 0, ..., T$,

$$c_t + b_{t+1} = Rb_t$$

$$m_t + \kappa_t = R\kappa_{t-1}$$

$$\kappa_t = R\kappa_{t-1} - m_t.$$

Finally, we have

$$b_{T+1} = \frac{\rho D}{\beta^T} c_T$$

$$\kappa_T = \frac{\rho D}{\beta^T} m_T.$$

We can solve this backwards.

$$\kappa_{t-1} = \frac{\kappa_t + m_t}{R},$$

where $\kappa_{-1} = 1$.

$$m_{t+1} = \frac{\frac{Q_{t+1}}{Q_t}\beta R m_t}{1 - \frac{\rho}{\beta^t}\left(1 - \frac{Q_{t+1}}{Q_t}\right)\frac{Dm_t}{\kappa_t}}$$

$$\frac{Q_{t+1}}{Q_t}\beta R m_t = m_{t+1}\left[1 - \frac{\rho}{\beta^t}\left(1 - \frac{Q_{t+1}}{Q_t}\right)\frac{Dm_t}{\kappa_t}\right]$$

$$\left[\frac{Q_{t+1}}{Q_t}\beta R + \frac{\rho D}{\beta^t}\left(1 - \frac{Q_{t+1}}{Q_t}\right)\frac{m_{t+1}}{\kappa_t}\right] m_t = m_{t+1}$$

Thus we have the iteration

$$\kappa_t = \frac{\kappa_{t+1} + m_{t+1}}{R}$$

$$m_t = \frac{m_{t+1}}{\frac{Q_{t+1}}{Q_t}\beta R + \frac{\rho D}{\beta^t}\left(1 - \frac{Q_{t+1}}{Q_t}\right)\frac{m_{t+1}}{\kappa_t}}$$

$$m_T + \kappa_T = R\kappa_{T-1}$$

$$\kappa_T = \frac{\rho D}{\beta^T} m_T.$$

$$\left(1 + \frac{\rho D}{\beta^T}\right) m_T = R\kappa_{T-1}$$

$$m_T = \frac{\beta^T R\kappa_{T-1}}{\beta^T + \rho D}$$

$$\kappa_T = \frac{\rho D R\kappa_{T-1}}{\beta^T + \rho D}$$

We have a system of $2T$ equations for the κ_t and m_t. Once we solve them, we have

$$D \ln b_0 + G = \sum_{t=0}^{T} \left\{ [Q_t \beta^t \ln(m_t b_0) + \rho(Q_t - Q_{t+1})[D \ln(\kappa_t b_0) + G]] \right\}$$

We need

$$D = \sum_{t=0}^{T} \left[Q_t \beta^t + \rho(Q_t - Q_{t+1}) D \right]$$

$$G = \sum_{t=0}^{T} \left\{ [Q_t \beta^t \ln(m_t) + \rho(Q_t - Q_{t+1})[D \ln(\kappa_t) + G]] \right\}$$

$$\left[1 - \sum_{t=0}^{T} \rho(Q_t - Q_{t+1}) \right] D = \sum_{t=0}^{T} Q_t \beta^t$$

$$D = \frac{\sum_{t=0}^{T} Q_t \beta^t}{1 - \sum_{t=0}^{T} \rho(Q_t - Q_{t+1})}$$

$$\begin{aligned} D &= \frac{\sum_{t=0}^{T} Q_t \beta^t}{1 - \rho \sum_{t=0}^{T} Q_t + \rho \sum_{t=0}^{T} Q_{t+1}} \\ &= \frac{\sum_{t=0}^{T} Q_t \beta^t}{1 - \rho \sum_{t=0}^{T} Q_t + \rho \sum_{i=1}^{T+1} Q_i} \\ &= \frac{\sum_{t=0}^{T} Q_t \beta^t}{1 - \rho Q_0 + \rho Q_{T+1}} \end{aligned}$$

$$D = \frac{\sum_{t=0}^{T} Q_t \beta^t}{1 - \rho Q_0}$$

If $Q_1 = Q_2 = \cdots = Q_T = 1$, then

$$D = \frac{\sum_{t=0}^{T} \beta^t}{1 - \rho} = \frac{1 - \beta^{T+1}}{(1 - \rho)(1 - \beta)}.$$

The m_t equation simplifies to

$$m_{t+1} = \beta R m_t.$$

Since there is no uncertainty, we need $\kappa_T = 1$ because b_{T+1} has to equal b_0. Therefore, since

$$m_t = (\beta R)^t m_0.$$

$$1 = \kappa_T = \frac{\rho D}{\beta^T} m_T = \frac{\rho D}{\beta^T} (\beta R)^T m_0 = \rho R^T m_0 D$$

By the Envelope Theorem,

$$v'(b_0) = \frac{\partial \mathcal{L}}{\partial b_0} = R\lambda_0 = \frac{R}{m_0 b_0}$$

But we have

$$v'(b_0) = \frac{D}{b_0}.$$

Thus

$$D = \frac{R}{m_0}.$$

Therefore,

$$1 = \rho R^{T+1}.$$

The situation is more complicated when we do have mortality risk though because then we have to account for the distribution of wealth.

In each period, measure 1 of households is born and measure 1 dies, so the population remains constant at

$$P = \sum_{t=0}^{T} Q_t.$$

Each household born has a probability $Q_t - Q_{t+1}$ of inheriting from a household of age t and draws b_0 from the distribution g_t. Note that what we would call B in the other models corresponds to Rb_0 here. Thus

$$E[b_0] = \sum_{t=0}^{T}(Q_t - Q_{t+1})E[b_{t+1}] = \sum_{t=0}^{T}(Q_t - Q_{t+1})\kappa_t E[b_0].$$

Thus in equilibrium we must satisfy the condition

$$1 = \sum_{t=0}^{T}(Q_t - Q_{t+1})\kappa_t. \tag{22}$$

For $T = 1$, we have

$$1 = [(1 - Q)\kappa_0 + Q\kappa_1]$$

Our equations are

$$1 = \frac{\kappa_0 + m_0}{R}$$

$$\kappa_0 = \frac{\kappa_1 + m_1}{R}$$

$$\kappa_0 = R - m_0$$

$$m_0 = \frac{m_1}{\beta QR + \rho D(1-Q)\frac{m_1}{\kappa_0}}$$

$$\kappa_1 = \frac{Dm_1\rho}{\beta}$$

$$\kappa_0 = \frac{1+\frac{D\rho}{\beta}}{R}m_1$$

$$\frac{m_1}{\kappa_0} = \frac{R}{1+\frac{D\rho}{\beta}} = \frac{\beta R}{\beta + D\rho}$$

$$\frac{\kappa_1}{\kappa_0} = \frac{D\rho}{\beta}\frac{R}{1+\frac{D\rho}{\beta}} = \frac{D\rho}{\beta + D\rho}R$$

$$m_0 = \frac{m_1}{\beta R\left[Q + \frac{\rho D(1-Q)}{\beta + D\rho}\right]} = \frac{m_1(\beta + D\rho)}{\beta R[\beta Q + D\rho Q + \rho D - \rho DQ]}$$

$$m_1 = \frac{\beta R[\beta Q + \rho D]m_0}{\beta + D\rho}$$

$$m_0 = \frac{\beta + D\rho}{\beta R(\beta Q + \rho D)}m_1$$

$$1 = \frac{m_0}{R} + \frac{m_1 + \kappa_1}{R^2}$$

$$1 = \frac{\beta + D\rho}{\beta R^2[\beta Q + \rho D]}m_1 + \frac{m_1}{R^2} + \frac{D\rho}{R^2\beta}m_1$$

$$\beta R^2 = \left[\frac{\beta + D\rho}{\beta Q + \rho D} + \beta + D\rho\right]m_1$$

$$\beta R^2 = (\beta + D\rho)\frac{1 + \beta Q + \rho D}{\beta Q + \rho D}m_1$$

$$m_1 = \frac{\beta R^2(\beta Q + \rho D)}{(\beta + \rho D)(1 + \beta Q + \rho D)}$$

$$m_0 = \frac{R}{1 + \beta Q + \rho D}$$

$$\kappa_0 = R - m_0 = R - \frac{R}{1 + \beta Q + \rho D} = \frac{\beta Q + \rho D}{1 + \beta Q + \rho D}R$$

$$\kappa_1 = \frac{Dm_1\rho}{\beta} = \frac{\rho D R^2(\beta Q + \rho D)}{(\beta + \rho D)(1 + \beta Q + \rho D)}$$

If $Q = 1$,

$$D = \frac{\sum_{t=0}^{T} Q_t\beta^t}{1-\rho} = \frac{1+\beta}{1-\rho}$$

$$1+\beta Q+\rho D = 1+\beta+\frac{1+\beta}{1-\rho} = (1+\beta)\left(1+\frac{1}{1-\rho}\right) = \frac{1+\beta}{1-\rho}$$

$$m_0 = \frac{1-\rho}{1+\beta}R$$

$$m_1 = \frac{\beta R^2(\beta+\rho D)}{(\beta+\rho D)(1+\beta Q+\rho D)} = \frac{\beta R^2}{\frac{1-\rho}{1+\beta}} = \beta R\frac{1+\beta}{1-\rho}R = \beta R m_0$$

$$\begin{aligned}\kappa_0 &= \frac{\beta+\rho D}{1+\beta+\rho D}R\\ &= \frac{\beta+\frac{\rho(1+\beta)}{1-\rho}}{\frac{1+\beta}{1-\rho}}R = \frac{(1-\rho)\beta+\rho+\rho\beta}{1+\beta}R = \frac{\beta+\rho}{1+\beta}R\end{aligned}$$

$$\kappa_1 = \frac{\rho D R^2(\beta Q+\rho D)}{(\beta+\rho D)(1+\beta Q+\rho D)} = \frac{\rho D R^2}{1+\beta Q+\rho D} = \frac{\rho R^2\frac{1+\beta}{1-\rho}}{\frac{1+\beta}{1-\rho}} = \rho R^2.$$

In order for $\kappa_1 = 1$, we need $\rho R^2 = 1$ as above.

For general Q, we need

$$\kappa_0 = \frac{\beta Q+\rho D}{1+\beta Q+\rho D}R$$

$$\kappa_1 = \frac{\rho D R^2(\beta Q+\rho D)}{(\beta+\rho D)(1+\beta Q+\rho D)}$$

$$D = \frac{1+\beta Q}{1-\rho}$$

$$1+\beta Q+\rho D = 1+\beta Q+\rho\frac{1+\beta Q}{1-\rho} = (1+\beta Q)\left[\frac{1-\rho+\rho}{1-\rho}\right] = \frac{1+\beta Q}{1-\rho}$$

$$\beta Q+\rho D = \frac{1+\beta Q}{1-\rho}-1 = \frac{1+\beta Q-1+\rho}{1-\rho} = \frac{\rho+\beta Q}{1-\rho}$$

$$\kappa_0 = \frac{\rho+\beta Q}{1+\beta Q}R$$

$$\beta+\rho D = \frac{\beta(1-\rho)+\rho(1+\beta Q)}{1-\rho} = \frac{\beta+\rho-\beta\rho+\rho\beta Q}{1-\rho}$$

$$\kappa_1 = \frac{\rho\frac{1+\beta Q}{1-\rho}\frac{\rho+\beta Q}{1-\rho}}{\frac{\beta+\rho-\beta\rho+\rho\beta Q}{1-\rho}\frac{1+\beta Q}{1-\rho}}R^2$$

$$\kappa_1 = \frac{\rho(\rho+\beta Q)}{\beta+\rho-\beta\rho+\rho\beta Q}R^2$$

$$1 = (1-Q)\frac{\rho+\beta Q}{1+\beta Q}R + \frac{\rho Q(\rho+\beta Q)}{\beta+\rho-\beta\rho+\rho\beta Q}R^2$$

Suppose that $Q = 1-h$ where h is small. To first order in h, we will have

$$1 = h\frac{\rho+\beta}{1+\beta}R + \frac{\rho(1-h)(\rho+\beta(1-h))}{\beta+\rho-\beta\rho+\rho\beta(1-h)}R^2 + O(h^2).$$

$$1 = h\frac{\rho+\beta}{1+\beta}R + \frac{\rho(1-h)(\rho+\beta-\beta h)}{\beta+\rho-\rho\beta h}R^2 + O(h^2).$$

$$1 = h\frac{\rho+\beta}{1+\beta}R + \frac{\rho(1-h)\left(1-\frac{\beta}{\rho+\beta}h\right)}{1-\frac{\rho\beta}{\rho+\beta}h}R^2 + O(h^2).$$

$$1 = h\frac{\rho+\beta}{1+\beta}R + \left[1-h-\frac{\beta}{\rho+\beta}h+\frac{\rho\beta}{\rho+\beta}h\right]\rho R^2 + O(h^2).$$

$$1 = h\frac{\rho+\beta}{1+\beta}R + \left[1-\frac{\rho+2\beta-\rho\beta}{\rho+\beta}h\right]\rho R^2 + O(h^2).$$

Let

$$R = \rho^{-1/2}\left[1+r_1 h\right] + O(h^2)$$

$$1 = h\frac{\rho+\beta}{1+\beta}\rho^{-1/2}(1+r_1h) + \left[1-\frac{\rho+2\beta-\rho\beta}{\rho+\beta}h\right](1+2r_1h) + O(h^2)$$

$$0 = h\frac{\rho+\beta}{1+\beta}\rho^{-1/2} - \frac{\rho+2\beta-\rho\beta}{\rho+\beta}h + 2r_1h + O(h^2)$$

$$r_1 = \frac{1}{2}\left[\frac{\rho+2\beta-\rho\beta}{\rho+\beta} - \frac{\rho+\beta}{1+\beta}\rho^{-1/2}\right]$$

For $T=2$, Mathematica finds there is still a unique solution, but it may not be true that we have a linear system for general T. For $T=2$, we have six equations linear in the six ratios $\frac{\kappa_1}{\kappa_0}, \frac{\kappa_2}{\kappa_1}, \frac{m_1}{\kappa_0}, \frac{m_2}{\kappa_0}, \frac{m_1}{m_0}, \frac{m_2}{m_1}$. But for $T=3$, we have eight equations and 9 such ratios, and Mathematica did not find an analytic solution.

The system generalizes as follows.

$$m_t = \frac{m_{t+1}}{\frac{Q_{t+1}}{Q_t}\beta R + \frac{\rho D}{\beta^t}\left(1-\frac{Q_{t+1}}{Q_t}\right)\frac{m_{t+1}}{\kappa_t}}$$

$$\kappa_t = \frac{\kappa_{t+1}+m_{t+1}}{R}$$

$$\kappa_T = \frac{\rho D}{\beta^T}m_T.$$

$$\left(1+\frac{\rho D}{\beta^T}\right)m_T = R\kappa_{T-1}$$

$$m_T = \frac{\beta^T R\kappa_{T-1}}{\beta^T+\rho D}$$

$$m_{T-1} = \frac{m_T}{\frac{Q_T}{Q_{T-1}}\beta R + \frac{\rho D}{\beta^{T-1}}\left(1-\frac{Q_T}{Q_{T-1}}\right)\frac{\beta^T R}{\beta^T+\rho D}}$$

$$m_{T-1} = \frac{1}{\beta R}\frac{m_T}{\frac{Q_T}{Q_{T-1}} + \left(1-\frac{Q_T}{Q_{T-1}}\right)\frac{\rho D}{\beta^T+\rho D}}$$

$$\frac{m_{t+1}}{\kappa_t} = R - \frac{\kappa_{t+1}}{\kappa_t} = R - \frac{\kappa_{t+2}+m_{t+2}}{\kappa_{t+1}+m_{t+1}}$$

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A Properties of Solution to the Household Problem

Let us define

$$\begin{aligned}\mathcal{C} &= \{(c_0, ..., c_T) \in \mathbf{R}_+^{T+1} \\ &: (\exists (b_0, ..., b_{T+1}) \in \{0\} \times \mathbf{R}_+^T : (\forall t \in \{0, ..., T\})[c_t + b_{t+1} = y_t + Rb_t]\}\end{aligned}$$

to be the feasible set of consumption paths. Is $\mathcal{C}$ convex? Let $c^1, c^2 \in \mathcal{C}$ and $\lambda \in [0,1]$. Let

$$c = \lambda c^1 + (1-\lambda)c^2.$$

Then for $t = 0, ..., T$, $c_t^1, c_t^2 \geq 0$. Then

$$c_t = \lambda c_t^1 + (1-\lambda)c_t^2 \geq 0,$$

so $c \in \mathbf{R}_+^{T+1}$. Let $b^1, b^2 \in \{0\} \times \mathbf{R}_+^{T+1}$ such that $c_t^i + b_{t+1}^i = we_t + Rb_t^i$ for $t = 0, ..., T$. Let $b = \lambda b^1 + (1-\lambda)b^2$. Then

$$b_0 = \lambda b_0^1 + (1-\lambda)b_0^2 = \lambda(0) + (1-\lambda)(0) = 0.$$

For $t = 1, ..., T$,

$$b_t = \lambda b_t^1 + (1-\lambda)b_t^2 \geq 0.$$

Therefore, $b \in \{0\} \times \mathbf{R}_+^{T+1}$. For $t \in \{0, ..., T\}$,

$$\begin{aligned}
\lambda(c_t^1 + b_{t+1}^1) + (1-\lambda)(c_t^2 + b_{t+1}^2) &= \lambda(y_t + Rb_t^1) + (1-\lambda)(y_t + Rb_t^2) \\
\lambda c_t^1 + (1-\lambda)c_t^2 + \lambda b_{t+1}^1 + (1-\lambda)b_{t+1}^2 &= y_t + R[\lambda b_t^1 + (1-\lambda)b_t^2] \\
c_t + b_{t+1} &= y_t + Rb_t.
\end{aligned}$$

Thus $c \in \mathcal{C}$, so $\mathcal{C}$ is convex.

For $c \in \mathcal{C}$,

$$b_{s+1} - Rb_s = y_s - c_s$$

$$\sum_{s=0}^{t} \frac{b_{s+1} - Rb_s}{R^s} = \sum_{s=0}^{t} \frac{y_s - c_s}{R^s}$$

$$\sum_{s=0}^{t} \frac{b_{s+1}}{R^s} - \sum_{s=0}^{t} \frac{b_s}{R^{s-1}} = \sum_{s=0}^{t} \frac{y_s - c_s}{R^s}$$

$$\frac{b_{t+1}}{R^t} + \sum_{s=0}^{t-1} \frac{b_{s+1}}{R^s} - \sum_{s=1}^{t} \frac{b_s}{R^{s-1}} = \sum_{s=0}^{t} \frac{y_s - c_s}{R^s}$$

$$\frac{b_{t+1}}{R^t} + \sum_{s=0}^{t-1} \frac{b_{s+1}}{R^s} - \sum_{s=0}^{t-1} \frac{b_{s+1}}{R^s} = \sum_{s=0}^{t} \frac{y_s - c_s}{R^s}$$

$$b_{t+1} = \sum_{s=0}^{t} \frac{y_s - c_s}{R^{s-t}}.$$

For $c \in \mathcal{C}$,

$$\begin{aligned}
U(c) &= \sum_{t=0}^{T} [\beta^t Q_t u(c_t) + D_{t+1} u(b_{t+1})] \\
&= \sum_{i=0}^{T} \left[\beta^i Q_t u(c_i) + D_{i+1} u\left(\sum_{j=0}^{i} \frac{y_j - c_j}{R^{j-i}} \right) \right]
\end{aligned}$$

Let

$$\Theta_{ij} = \begin{cases} 1 & i \geq j \\ 0 & i < j \end{cases}$$

$$\begin{aligned} \frac{\partial U}{\partial c_t} &= \sum_{i=0}^{T} \left[\beta^i Q_t u'(c_i) \delta_{it} - \frac{D_{i+1}}{R^{t-i}} u' \left(\sum_{j=0}^{i} \frac{y_j - c_j}{R^{j-i}} \right) \Theta_{it} \right] \\ &= \beta^t Q_t u'(c_t) - \sum_{i=0}^{t} \frac{D_{i+1}}{R^{t-i}} u' \left(\sum_{j=0}^{i} \frac{y_j - c_j}{R^{j-i}} \right) \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 U}{\partial c_s \partial c_t} &= \beta^t Q_t u''(c_t) \delta_{ts} - \sum_{i=0}^{t} \frac{D_{i+1}}{R^{t-i}} u'' \left(\sum_{j=0}^{i} \frac{y_j - c_j}{R^{j-i}} \right) \left(-\frac{\Theta_{is}}{R^{s-i}} \right) \\ &= \beta^t Q_t u''(c_t) \delta_{ts} + \sum_{i=0}^{\min\{s,t\}} \frac{D_{i+1}}{R^{s+t-2i}} u'' \left(\sum_{j=0}^{i} \frac{y_j - c_j}{R^{j-i}} \right) \end{aligned}$$

This will definitely be negative definite if the D_t are small enough, in which case there will be a unique maximum.

Let us define

$$\mathcal{C}_0 = \{x \geq 0 : (\exists c \in \mathcal{C})[x = c_0]\}.$$

We want to show that $\mathcal{C}_0 = [0, c_0^{\max}]$ for some $c_0^{\max} \geq 0$.

$$\begin{aligned} \mathcal{C} &= \{(c_0, ..., c_T) \in \mathbf{R}_+^{T+1} \\ &: (\exists (b_0, ..., b_{T+1}) \in \{0\} \times \mathbf{R}_+^T : (\forall t \in \{0, ..., T\})[c_t + b_{t+1} = y_t + Rb_t]\} \end{aligned}$$

Let

$$\underline{c} = \left(0, 0, ..., 0, R^T \sum_{t=0}^{T} \frac{y_t}{R^t}\right) \in \mathbf{R}_+^{T+1}.$$

For $0 < t \leq T$,

$$\underline{b}_t = \sum_{s=0}^{t-1} \frac{y_s - \underline{c}_s}{R^{s-t+1}} = \sum_{s=0}^{t-1} \frac{y_s}{R^{s-t+1}} \geq 0$$

$$\underline{b}_{T+1} = \sum_{s=0}^{T} \frac{y_s - \underline{c}_s}{R^{s-T}} = \sum_{s=0}^{T} \frac{y_s}{R^{s-T}} - R^T \sum_{t=0}^{T} \frac{y_t}{R^t} = 0.$$

Thus $0 \in \mathcal{C}_0$.

Let $x > 0$ also be in $\mathcal{C}_0$ and let $z \in (0, x)$. Then there exists $c \in \mathbf{R}_+^{T+1}$ and $b^x \in \{0\} \times \mathbf{R}_+^T$

such that

$$c_t + b^x_{t+1} = y_t + Rb^x_t$$

for all $t = 0, ..., T$. Then

$$b^x_t = \sum_{s=0}^{t-1} \frac{y_s - c_s}{R^{s-t+1}}.$$

Let $d \in \mathbf{R}^{T+1}_+$ be such that $d_0 = z$, $d_t = c_t$ for $t = 1, ..., T-1$, and

$$d_T = c_T + R^T(x - z).$$

Let $b^y_0 = 0$. For $t = 1, ..., T+1$, let

$$b^y_t = \sum_{s=0}^{t-1} \frac{y_s - d_s}{R^{s-t+1}}.$$

For $0 < t \leq T$,

$$\begin{aligned} b^y_t &= \sum_{s=0}^{t-1} \frac{y_s - d_s}{R^{s-t+1}} = \frac{y_0 - z}{R^{1-t}} + \sum_{s=1}^{t-1} \frac{y_s - c_s}{R^{s-t+1}} \\ &= \frac{y_0 - c_0}{R^{1-t}} + \sum_{s=1}^{t-1} \frac{y_s - c_s}{R^{s-t+1}} + \frac{x - z}{R^{1-t}} = \sum_{s=0}^{t-1} \frac{y_s - c_s}{R^{s-t+1}} + \frac{x - z}{R^{1-t}} \\ &= b^x_t + \frac{x - z}{R^{1-t}} > b^x_t \geq 0. \end{aligned}$$

$$\begin{aligned} b^y_{T+1} &= \sum_{s=0}^{T} \frac{y_s - d_s}{R^{s-T}} = \frac{y_0 - z}{R^{-T}} + \sum_{s=1}^{T-1} \frac{y_s - c_s}{R^{s-T}} + y_T - d_T \\ &= \frac{y_0 - c_0}{R^{-T}} + \sum_{s=1}^{T-1} \frac{y_s - c_s}{R^{s-T}} + \frac{c_0 - z}{R^{-T}} + y_T - c_T - R^T(x - z) \\ &= \sum_{s=0}^{T} \frac{y_s - c_s}{R^{s-T}} + R^T[x - z - (x - z)] \\ &= b^x_{T+1} \geq 0. \end{aligned}$$

Thus

$$b^y \in \{0\} \times \mathbf{R}^T_+,$$

and

$$d_t + b^y_{t+1} = y_t + Rb^y_t$$

for $t = 0, ..., T$. Therefore, $d \in \mathcal{C}$. Since $d_0 = y$, $y \in \mathcal{C}_0$. Thus $\mathcal{C}_0$ is an interval bounded below by 0.

Let c^n be a sequence in $\mathcal{C}$ that converges to $c \in \mathbf{R}_+^{T+1}$. For each n, there exists $b^n \in \{0\} \times \mathbf{R}_+^T$ such that for all $t \in 0, ..., T$

$$c_t^n + b_{t+1}^n = y_t + Rb_t^n.$$

For $t = 1, ..., T+1$, we have

$$b_t^n = \sum_{s=0}^{t-1} \frac{y_s - c_s^n}{R^{s-t+1}} \geq 0$$

Then

$$b_t^n \longrightarrow b_t = \sum_{s=0}^{t-1} \frac{y_s - c_s}{R^{s-t+1}} \geq 0.$$

Meanwhile we define $b_0 = 0$. Then since $\{0\} \times \mathbf{R}_+^T$ is closed, $b \in \{0\} \times \mathbf{R}_+^T$. And for all $t = 0, ..., T$,

$$c_t + b_{t+1} = y_t + Rb_t.$$

Thus $c \in \mathcal{C}$, and $\mathcal{C}$ is closed. Note also that we have

$$\sum_{t=0}^{T} \frac{c_t}{R^t} + \frac{b_{T+1}}{R^t} = \sum_{t=0}^{T} \frac{y_t}{R^t},$$

so

$$0 \leq c_t \leq R^t \sum_{s=0}^{T} \frac{y_s}{R^s}.$$

Thus $\mathcal{C}$ is also compact.

Now suppose x_n is a sequence in $\mathcal{C}_0$ that converges to $x \geq 0$. For each n, there exists $c^n \in \mathcal{C}$ such that $x_n = c_0^n$. Since $\mathcal{C}$ is compact, there will be a convergent subsequence c^{m_n} that converges to $c \in \mathcal{C}$. Then $x_{m_n} \to c_0 \in \mathcal{C}_0$. Therefore, $x_n \to c_0 \in \mathcal{C}_0$. Thus $\mathcal{C}_0$ is closed and compact.