

Capital in a Segregated Economy*

Preliminary and Incomplete

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Abstract

As an alternative to massive heterogeneous-agent models, we introduce a new kind of model in which inequality is introduced exogenously, so the policy implications of severe wealth inequality can be analyzed more readily. This is a hybrid model in which households are segregated into two groups: laborers and capitalists. The laborers comprise the bulk of the population. Unable to pass on significant amounts to their children, they are described by a standard overlapping-generations model of perfectly competitive agents. In contrast, the handful of capitalists can fully indulge the pure altruism they feel to their children and are modeled as infinitely-lived dynasties. Contrary to the frequent characterization of capitalists as essential “job creators”, the welfare of laborers would decrease at most by the equivalent of 13% of consumption if price-taking capitalists were eliminated from the economy. However, this hybrid model also allows us to consider the case where the capitalists are price setters instead of price takers, which yields predictions shockingly different from perfectly competitive macro models, predictions that can, however, be reconciled with empirical data. Deviations from the Pareto efficient Euler equation are proportional to the fraction of wealth owned by capitalists and the curvature of the production function, both of which are quite high in the Twenty-First Century American economy, opening the door for government intervention. When we incorporate taxes in the model, we find that both capitalists and laborers would prefer to eliminate capital taxes in the long run. However, a Pareto-improving transition to this steady state requires capital taxes for capitalists to be raised initially.

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Thomas Piketty’s (2014) *Capital in the 21st Century* put a spotlight on the high levels of inequality that have been present in the global economy for as long as we have data. A major point of the book is that “the distribution of capital ownership (and of income from capital) is always more concentrated than the distribution of income from labor.”¹ Piketty documents the class conflict made famous by Karl Marx. Whereas a small fraction, between 1% and 0.1%, are capitalists who have accumulated more wealth than could be earned from a lifetime of labor and live off the return on this wealth, the bulk of the population gets most of their income by selling their labor.² One of the biggest challenges for macroeconomists is the construction of models that incorporate this fact. There is a huge literature that seeks to endogenously replicate the empirical distribution of income and wealth with elaborate heterogeneous-agent models, but the policy implications of these models have largely been an afterthought because it is so difficult to get the distributions right. Here I propose an alternative approach. Inequality can be introduced exogenously with a much simpler, segregated-economy model that takes seriously the distinction between capitalists and laborers.³ The simplicity of this model also allows us to examine distortions that are beyond the scope of heterogeneous-agent models in which diversity is primarily a consequence of accumulated shocks.

Although Marx is often credited with the idea of a class conflict between capitalists and workers, this is in fact a central theme of the *Wealth of Nations*, which was published nearly a century before *Capital* in 1776. Contrary to the popular view that Adam Smith “invented” capitalism, he was actually quite critical of capitalists and admonished the rest of us to consider any proposals by capitalists with a highly skeptical ear.⁴ Nevertheless, Marx and Smith offered radically different explanations for how capitalists “exploit” workers. Since Marx took it as axiomatic that marginal costs are constant, he inferred that any profit had to be stolen from workers, i.e. capitalists can only sustain themselves by exploiting their workers. Smith, on the other hand, argued that capitalists exert control by colluding with each other to reshape the marketplace so as to maximize their profits. He made especial note of how, *ceteris paribus*, capitalists and workers are caught in a zero-sum game. While under

¹p. 212.

²At the beginning of the 20th Century, the richest 1% lived off the earnings of their land or capital. Since then, the fraction living off their wealth has shrunk to about 0.1%. Inequality has been much stabler than this statistic might suggest, however, because there is much greater dispersion in labor incomes today.

³Several researchers have considered models in which capitalists and laborers behave differently. Judd (1985), for example, assumes that laborers cannot save and must live hand to mouth. The innovation here that is all the differences between capitalists and laborers could be explained endogenously in terms of widely disparate wealth levels. However, the source of these differences is not relevant to the questions at hand, and modeling them explicitly would explode the complexity of the computation.

⁴Smith never used the words capitalist or capitalism, which came into wide circulation later. If you define capitalism as the belief that private property generally promotes prosperity, it is fair to say that Smith invented capitalism. If, however, you define capitalism as an economic system where capitalists make all decisions about the production and distribution of goods, Smith was very much against that.

perfect competition neither side actually plays this game since the outcome is determined by equilibrium conditions, Smith did not view this conflict under the lens of perfect competition. He clearly envisioned it as a game that the capitalist invariably wins. In a segregated-economy model it is straightforward to incorporate this “Smith distortion” by treating the capitalists as price setters instead of price takers.

A segregated economy is distinguished from a split economy (Feigenbaum (2015)) in that the two types of agents still interact through markets whereas in a split economy they would not interact at all—there would be two separate economies. The workers are modeled as typical lifecycle agents. We will assume they live for two periods, working in the first period and being retired in the second. Then there are capitalists who are modeled as infinite-horizon agents and who only earn income from capital.⁵ But there are many more laborers than capitalists, so the laborers will always be treated as price-taker while the capitalists are able to coordinate their actions.⁶ We will consider both the case where the capitalists are price-takers and where they are price-settlers. In the latter case, when they choose their saving, they will account for the effect their saving will have on the wages and saving of the laborers.

Kotlikoff and Summers (1981) showed the important link between bequests and wealth accumulation, but our understanding of this link has been stymied by our poor understanding of how to model bequests. If all households exhibit pure altruism as in Barro (1974), based on the amount of saving that we observe we would have to conclude that most parents do not actually care much for their children. Using a common estimate that the rate of return on capital is roughly 4%, a pure altruism model would require that parents value their children’s utility 70% less than their own. Thus most research has focused on models of impure altruism, as in De Nardi (2004), where bequests are a luxury good that only the rich can afford. Yet even if we assume impure altruism, it is still difficult to construct heterogeneous-agent models with a wealth distribution as concentrated as what we observe empirically (De Nardi and Yang (2014)).⁷ With the segregated economy model, we follow an approach informed by the intrafamily bargaining model of Feigenbaum and Li (2018). If parents and adult children both exhibit altruism toward each other, a net transfer from parents to children will only occur if there is a large wealth differential between the two parties. In very rich families where the parents control the purse strings, the

⁵Berg, Buﬃe, and Zanna (2018) uses a similar framework to study the effects of robotization with high-skilled workers and capitalists who live forever and low-skilled workers who do not save.

⁶In addition to their disparate numbers, perhaps a more compelling reason why capitalists are better able to exploit their pricing power is that it will always be optimal for them to reduce their saving from what they would do as price-takers. For a laborer who lives more than two periods, there will be incentives both to oversave and undersave, and the relative magnitudes will vary over the lifecycle, which will make it all the more difficult for laborers to coordinate their behavior.

⁷The “capitalist spirit” model of Carroll (2000) can easily achieve such an empirical wealth distribution by putting wealth into the utility function, but these ad hoc preferences are philosophically difficult to justify.

family should behave jointly like a dynasty with a plan for wealth to pass from one generation to the next. In more typical families, the parents and children will make economic decisions independently. On net, any gift-giving will be unintended since neither side will want to take from the other. A segregated economy model captures these two distinct behaviors. As a bonus, we can easily incorporate what Adam Smith would have said was the most crucial thing: the ability of a handful of capitalists to combine against the rest of the population.

Just as in the standard Cass (1965)-Koopmans (1965)-Ramsey (1928) (CKR) framework, the natural intuition that an infinitely lived dynasty of capitalists will seek to maximize their steady-state consumption is wrong if we account for Smith distortions. Likewise, the common perception that capitalists single-mindedly want to increase their wealth is incorrect too. Instead, as in the CKR framework, a capitalist will have to make a tradeoff between his current consumption and his long-term happiness. The bigger the generational discount rate, the more weight he will give to his current consumption. Where the segregated economy model deviates from the CKR framework is that the choices the capitalist makes are not Pareto optimal. In the steady state it is efficient for a capitalist dynasty to treat its capital like an annuity, consuming only the return at a rate equal to the dynasty's generational discount rate without depleting the principal. A price-taking capitalist will do this in a competitive equilibrium, but a price-setting capitalist will consume at a higher rate in the steady state.⁸ For the capitalist this has the advantage that he needs less capital to support his consumption, so he can consume more in the short run. For workers, this arrangement means there is less capital to support the population as a whole.

If the top 0.1% are capitalists who own 20% of the capital stock, their price-setting behavior will lower GDP by 7.5%, resulting in a welfare loss by the other 99.9% that is equivalent to the loss of 6% of consumption. In the present stylized model, we abstract from growth so price-setting leads to an actual decrease in output. More realistically, it would lead to a retardation of the growth rate. As much as we talk about inequality today, it is actually rather puzzling that the very wealthy do not have a much larger share of the capital stock. Indeed, the share of wealth owned by the 1% has barely increased from its nadir of 30% in the 1960s. At the beginning of the 20th Century, the 1% owned roughly 50% of the capital stock, but those 1% were primarily landowners and not capitalists.

Even if capitalists are price-takers though, their contribution to the economy is not as big as they are often given credit for. If the top 0.1% own 20% of the capital stock, the welfare of laborers would only decrease by the equivalent of 8% of consumption if those capitalists were eliminated from the economy. Even if the capitalists own 30% of the capital stock, the welfare of laborers would still only decrease by 13%.

Kuznets (1955) speculated that there was an inverted-U relationship between inequality and the size of an economy. As an economy starts to grow, inequality rises along with this growth, but eventually inequality reaches a zenith and

⁸This means that with price-setting capitalists, the estimate we get from the return on capital of how much they value their children will be biased downward.

further growth leads to a fall in inequality. As Piketty (2014) documented more than half a century later, the decreasing part of the inverted U now appears to have been a historical artifact. Kuznets was writing shortly after the end of one of the most turbulent eras in human history with the Great Depression and two world wars. There was considerable capital destruction, which led to a decrease in wealth inequality. Since the 1960s though, inequality and GDP have grown in sync. The segregated economy model predicts that destruction of the capital stock will result in a decrease in wealth inequality either if capitalists are price takers or price setters.

The present uncertainty in our measurement of even the most basic of macro-economic observables provides a lot of room to mask inefficient behavior that is beneficial to a small fraction of the population while causing welfare losses to the other 99+% larger than issues that receive much more complaints such as the business cycle (Lucas (2003)) and illegal immigration (Feigenbaum (2014)).

The advantage of a segregated economy model over the standard heterogeneous-agent approach where everyone is described by an overlapping-generations model is that it can very easily give us answers to how rising wealth inequality impacts policy. However, since inequality is exogenous in a segregated economy, the existing literature on inequality is still important for understanding how inequality arises and evolves. The two approaches should be viewed as complements rather than as rivals.

As an example, consider the question of the optimal mix of capital and labor taxes. There is a huge bifurcation of the literature regarding the wisdom of taxing capital. With the infinite-horizon models that used to be the norm in macroeconomics, there is a fairly robust finding that capital taxes should be set to zero (Judd (1985)) and Chamley (1986)). In overlapping-generations models, results are more mixed (Conesa, Kitao, and Krueger (2009)). In the extreme case of a two-period overlapping-generations model, it is the tax on labor income that should usually be set to zero unless that would require a prohibitively high tax on capital.

In the baseline calibration of the segregated-economy model, we find that Judd's result that it is optimal not to tax capital still holds. However, the gains from not taxing capital are almost entirely one-sided. The capitalists increase their consumption by 75% while the workers see an increase in welfare equivalent to 0.4% of current consumption. The sticking point is the transition. If capital taxes are simply eliminated and labor taxes are raised to balance the budget, the current generation of workers will suffer a loss equivalent to 4% of current consumption. The following generation will lose 5% of current consumption. It takes three generations for increases in the capital stock to result in higher after-tax wages. There is a Pareto-improving path to the no-capital-tax steady state, but this involves increasing the tax on capital for capitalists in the short run, which essentially zeros out their welfare gains.

The paper is organized as follows. The segregated economy model is introduced in Section 1.

1 A Segregated-Economy Model

A segregated-economy model is an overlapping-generations model with two types of individuals: capitalists (c) and laborers (l). Time is discrete. In each period a measure μ of laborers is born while the measure of capitalists is $1 - \mu$. In this simplest version of the model, laborers live for two periods. While young they work, and while old they live off their saving. Thus a period corresponds to the working life of a laborer, say thirty years, and the total population is $1 + \mu$.

A laborer born at t chooses his consumptions $\{c_{t,0}^l, c_{t+1,1}^l\}$ and leisure l_t to maximize his lifetime utility

$$U_t^l = \max u^l(c_{t,0}^l, l_t) + \beta_l u^l(c_{t+1,1}^l, 1) \quad (1)$$

subject to

$$c_{t,0}^l + k_{t+1}^l = (1 - \tau_t^l)W_t(1 - l_t) \quad (2)$$

$$c_{t+1,1}^l = (1 + (1 - \tau_{t+1}^{k,l})r_{t+1})k_{t+1}^l, \quad (3)$$

where k_{t+1}^l is the saving of a laborer at t . The government imposes taxes τ_t^l on labor income and $\tau_k^{k,l}$ on the capital income of laborers. The wage W_t and the before-tax return on capital r_t will be treated as given by the laborer.

Meanwhile, the capitalists live forever. They should be viewed as an infinitely-lived dynasty, and their population at t is the current generation of decision-making patriarchs. Given the dynasty's current capital k_t ,⁹ a capitalist at t maximizes¹⁰

$$U_t^c = \sum_{s=0}^{\infty} \beta_c^s u^c(c_{t+s}^c) \quad (4)$$

subject to

$$c_t^c + k_{t+1}^c = (1 + (1 - \tau_t^{k,c})r_t)k_t \quad (5)$$

and the no-Ponzi condition

$$\lim_{s \rightarrow \infty} \frac{k_{t+s+1}}{\prod_{i=0}^s (1 + (1 - \tau_{t+i}^{k,c})r_{t+i})} \geq 0. \quad (6)$$

Note that the capital income of capitalists may be taxed at a separate rate $\tau_t^{k,c}$ from the rate $\tau_t^{k,l}$ that it is taxed at for laborers. Before we finish describing the capitalist's problem, we need to specify the rest of the environment.

⁹Since k_t^c will serve as a state variable and appears far more often than its counterpart k_t^l , we suppress the superscript c .

¹⁰Given that there is only one type of labor in the model, capitalists would endogenously choose not to work for the calibrations of the model that we will consider. We exogenously impose that they do not work to streamline the model.

Let D_t constitute government debt. Then the capital stock is

$$K_t = \mu k_t^l + (1 - \mu)k_t - D_t \quad (7)$$

and the labor supply is

$$N_t = \mu(1 - l_t). \quad (8)$$

There is a constant returns to scale production function $F(K, N)$ such that

$$W_t = W(K_t, N_t) = F_N(K_t, N_t) \quad (9)$$

and

$$r_t = r(K_t, N_t) = F_K(K_t, N_t) - \delta, \quad (10)$$

where $\delta > 0$ is the depreciation rate. It is helpful to denote the after-tax gross return on capital (for capitalists) as

$$R_t = R_t(K_t, N_t) = 1 + (1 - \tau_t^{k,c})r(K_t, N_t). \quad (11)$$

The taxes, and possibly debt, are used to finance government expenditures of goods G_t . The government must satisfy its budget constraint

$$G_t + (1 + r_t)D_t = \tau_t^l W_t N_t + r_t(\mu \tau_t^{k,l} k_t^l + (1 - \mu) \tau_t^{k,c} k_t) + D_{t+1}. \quad (12)$$

We will consider two formulation of the equilibrium: one where the capitalists are price-takers and one where they are price-setters.

1.1 Price-Taking Formulation

In the first, the capitalists behave as price-takers, as is normally assumed in general equilibrium models. The capitalist's problem can then be described in terms of the Bellman equation

$$V_t^{PT}(k_t, K_t) = \max_{c_t^c, k_{t+1}} u^c(c_t^c) + \beta_c V_{t+1}^{PT}(k_{t+1}, K_{t+1}) \quad (13)$$

subject to the constraint

$$c_t^c + k_{t+1} = R_t(K_t, \tilde{N}_t(K_t))k_t, \quad (14)$$

where

$$K_{t+1} = \tilde{K}_{t+1}(k_t, K_t). \quad (15)$$

The maps $\tilde{K}_{t+1}(k_t, K_t)$ and $\tilde{N}_t(K_t)$ of capital and labor respectively will be determined in equilibrium.

A laborer born at t solves his problem given the factor prices W_t and r_{t+1} . Let $l_t(W_t, r_{t+1})$ and $k_{t+1}^l(W_t, r_{t+1})$ be the resulting leisure and saving of the

laborer.¹¹ Then a price-taking equilibrium will consist of a sequence of capitalist policy functions $c_t^c(k_t, K_t)$ and $k_{t+1}(k_t, K_t)$, a sequence of laborer policy functions $l_t(W_t, r_{t+1})$ and $k_{t+1}^l(W_t, r_{t+1})$, capital and labor maps $\tilde{K}_{t+1}(k_t, K_t)$ and $\tilde{N}_t(K_t)$, initial state variables k_0 and K_0 , and a sequence of government policy variables $\{G_t, D_t, \tau_t^l, \tau_t^{k,l}, \tau_t^{k,c}\}$ such that (i) $l_t(W_t, r_{t+1})$ and $k_{t+1}(k_t, K_t)$ solve the laborer's problem (1) given the factor prices W_t and r_{t+1} and the taxes τ_t^l and $\tau_t^{k,l}$; (ii) $c_t^c(k_t, K_t)$ and $k_{t+1}(k_t, K_t)$ solve the price-taking capitalist's problem (13) given the maps $\tilde{K}_{t+1}(k_t, K_t)$ and $\tilde{N}_t(K_t)$ and the tax $\tau_t^{k,c}$; (iii) the capital and labor maps satisfy the equations of motion

$$\begin{aligned} \tilde{K}_{t+1}(k_t, K_t) &= \mu k_{t+1}^l \left(W_t(K_t, \tilde{N}_t(K_t)), r_{t+1}(\tilde{K}_{t+1}(K_t), \tilde{N}_{t+1}(\tilde{K}_{t+1}(K_t))) \right) \\ &\quad + (1 - \mu) k_{t+1}(k_t, K_{t+1}) \end{aligned} \quad (16)$$

and

$$\tilde{N}_t(K_t) = \mu \left[1 - l_t \left(W_t(K_t, \tilde{N}_t(K_t)), r_{t+1}(\tilde{K}_{t+1}(K_t), \tilde{N}_{t+1}(\tilde{K}_{t+1}(K_t))) \right) \right]; \quad (17)$$

and (iv) the sequences $\{K_{t+1}, k_{t+1}, N_t, r_t, W_t, k_{t+1}^l\}_{t=0}^\infty$ generated by the functions $\tilde{K}_{t+1}(k_t, K_t)$, $k_{t+1}(k_t, K_t)$, $\tilde{N}_t(K_t)$, $r_t(K_t, N_t)$, $W_t(K_t, N_t)$, and $k_{t+1}^l(W_t, r_{t+1})$ and the government policy variables $\{G_t, D_t, \tau_t^l, \tau_t^{k,l}, \tau_t^{k,c}\}_{t=0}^\infty$ all satisfy the government budget constraint (12) and the government's no-Ponzi condition

$$\lim_{t \rightarrow \infty} \frac{D_{t+1}}{\prod_{i=0}^{\infty} (1 + r_i)} \leq 0. \quad (18)$$

1.2 Price-Setting Formulation

In the price-setting formulation, the capitalist's problem is described in terms of the Bellman equation

$$V_t^{PS}(k_t, K_t) = \max_{c_t^c, k_{t+1}} u^c(c_t^c) + \beta_c V_{t+1}^{PS}(k_{t+1}, K_{t+1}) \quad (19)$$

subject to the budget constraint (14) and the capital equation of motion

$$K_{t+1} = \mu k_{t+1}^l \left(W_t(K_t, \tilde{N}_t(k_t, K_t)), r_{t+1}(K_{t+1}, \tilde{N}_{t+1}(K_{t+1})) \right) + (1 - \mu) k_{t+1}. \quad (20)$$

The labor map $\tilde{N}_t(k_t, K_t)$ will be determined in equilibrium.

A price-taking equilibrium will consist of a sequence of capitalist policy functions $c_t^c(k_t, K_t)$, $k_{t+1}(k_t, K_t)$, $K_{t+1}(k_t, K_t)$; a sequence of laborer policy functions $l_t(W_t, r_{t+1})$ and $k_{t+1}^l(W_t, r_{t+1})$; a labor map $\tilde{N}_t(K_t)$, initial state variables

¹¹Much of the complexity in the equilibrium definitions comes from the possibility that the labor supply might depend on the future return of capital. For the specific preferences that we will consider, the labor supply will end up being independent of factor prices, which will greatly simplify the problem of finding an equilibrium.

k_0 and K_0 ; and a sequence of government policy variables $\{G_t, D_t, \tau_t^l, \tau_t^{k,l}, \tau_t^{k,c}\}$ such that (i) $l_t(W_t, r_{t+1})$ and $k_{t+1}(W_t, r_{t+1})$ solve the laborer's problem (1) given the factor prices W_t and r_{t+1} and the taxes τ_t^l and $\tau_t^{k,l}$; (ii) $c_t^c(k_t, K_t)$, $k_{t+1}(k_t, K_t)$, and $K_{t+1}(k_t, K_t)$ solve the price-setting capitalist's problem (19) given the map $\tilde{N}_t(K_t)$ and the tax $\tau_t^{k,c}$; (iii) the capital map satisfies

$$\tilde{N}_t(k_t, K_t) = \mu \left[1 - l_t \left(W_t(K_t, \tilde{N}_t(k_t, K_t)), r_{t+1}(K_{t+1}(k_t, K_t), \tilde{N}_{t+1}(k_{t+1}(k_t, K_t), K_{t+1}(k_t, K_t))) \right) \right]; \quad (21)$$

and (iv) the sequences $\{K_{t+1}, k_{t+1}, N_t, r_t, W_t, k_{t+1}^l\}_{t=0}^\infty$ generated by the functions $K_{t+1}(k_t, K_t)$, $k_{t+1}(k_t, K_t)$, $\tilde{N}_t(k_t, K_t)$, $r_t(K_t, N_t)$, $W_t(K_t, N_t)$, and $k_{t+1}^l(W_t, r_{t+1})$ and the government policy variables $\{G_t, D_t, \tau_t^l, \tau_t^{k,l}, \tau_t^{k,c}\}_{t=0}^\infty$ all satisfy the government budget constraint (12) and the government's no-Ponzi condition (18).

2 The Solution to the Laborer's Problem

The laborer's problem is the same for both the price-setting and price-taking equilibria. We specialize to the case where

$$u^c(c, l) = \eta \ln c + (1 - \eta) \ln l.$$

3 The Price-Taking Equilibrium

Note that we get the standard Euler equation

$$u'(c_t) = \beta_c R_{t+1} u'(c_{t+1}).$$

In a steady state, this simplifies to the condition that $R_{PT}^* = \beta_c^{-1}$, which makes very strong—empirically speaking, overly strong—predictions.¹²

For a capitalist, the lifetime budget constraint is

$$\sum_{s=0}^{\infty} \frac{c_{t+s}^c}{R_{t,s}} \leq (1 + (1 - \tau_t^k) r_t) k_t^c \quad (22)$$

For laborers, saving is

$$k_{t+1}^l = \frac{c_{t+1,1}^l - (1 - \tau_{t+1}^l) W_{t+1} e_1 (1 - l_{t+1,1})}{1 + (1 - \tau_{t+1}^k) r_{t+1}}. \quad (23)$$

¹²In this paper we have abstracted from growth. If we had growth, the steady-state Euler equation would impose that R_{PT}^* is a function of β_c and other preference parameters. This will not change the point that the condition makes an overly strong prediction.

Inserting this into the young laborer's budget constraint, we get the laborer's lifetime budget constraint

$$c_{t,0}^l + (1 - \tau_t^l)W_t e_0 l_{t,0} + \frac{c_{t+1,1}^l + (1 - \tau_{t+1}^l)W_{t+1} e_1 l_{t+1,1}}{1 + (1 - \tau_{t+1}^k)r_{t+1}} = (1 - \tau_t^l)W_t e_0 + \frac{(1 - \tau_{t+1}^l)W_{t+1} e_1}{1 + (1 - \tau_{t+1}^k)r_{t+1}} \quad (24)$$

In an equilibrium, a laborer at t solves his problem given the prices W_t , W_{t+1} , and r_{t+1} . A capitalist at t solves for the sequences $\{k_{t+s+1}^c, r_{t+s+1}\}_{s=0}^{\infty}$ given the market-clearing constraints and the laborer's policy functions. Thus the capitalist is *not* a price-taker.

Let us define

$$v^l(E, w) = \max u^l(c, l) \quad (25)$$

subject to

$$c + wl = E \quad (26)$$

and

$$0 \leq l \leq 1. \quad (27)$$

Then we can rewrite the laborer's problem as

$$U_t^l = \max v^l(E_{t,0}^l, (1 - \tau_t^l)W_t e_0) + \beta v^l(E_{t+1,1}^l, (1 - \tau_{t+1}^l)W_{t+1} e_1) \quad (28)$$

subject to

$$E_{t,0}^l + \frac{E_{t+1,1}^l}{1 + (1 - \tau_{t+1}^k)r_{t+1}} = (1 - \tau_t^l)W_t e_0 + \frac{(1 - \tau_{t+1}^l)W_{t+1} e_1}{1 + (1 - \tau_{t+1}^k)r_{t+1}}. \quad (29)$$

Then saving is

$$k_{t+1}^l = (1 - \tau_t^l)W_t e_0 - E_{t,0}^l. \quad (30)$$

The intertemporal Lagrangian is

$$\begin{aligned} \mathcal{L}_t^l &= v^l(E_{t,0}^l, (1 - \tau_t^l)W_t e_0) + \beta v^l(E_{t+1,1}^l, (1 - \tau_{t+1}^l)W_{t+1} e_1) + \\ &\lambda_t \left[(1 - \tau_t^l)W_t e_0 + \frac{(1 - \tau_{t+1}^l)W_{t+1} e_1}{1 + (1 - \tau_{t+1}^k)r_{t+1}} - E_{t,0}^l - \frac{E_{t+1,1}^l}{1 + (1 - \tau_{t+1}^k)r_{t+1}} \right] \end{aligned} \quad (31)$$

The first-order conditions are

$$\frac{\partial \mathcal{L}_t^l}{\partial E_{t,0}^l} = \frac{\partial v^l}{\partial E}(E_{t,0}^l, (1 - \tau_t^l)W_t e_0) - \lambda_t = 0$$

$$\frac{\partial \mathcal{L}_t^l}{\partial E_{t+1,1}^l} = \beta \frac{\partial v^l}{\partial E}(E_{t+1,1}^l, (1 - \tau_{t+1}^l)W_{t+1} e_1) - \frac{\lambda_t}{1 + (1 - \tau_{t+1}^k)r_{t+1}} = 0.$$

Thus the laborer's Euler equation is

$$\frac{\partial v^l}{\partial E}(E_{t,0}^l, (1 - \tau_t^l)W_t e_0) = \beta \left[1 + (1 - \tau_{t+1}^k)r_{t+1} \right] \frac{\partial v^l}{\partial E}(E_{t+1,1}^l, (1 - \tau_{t+1}^l)W_{t+1} e_1). \quad (32)$$

Once we have solved the laborer's problem, we can express his policy variables as functions of K_t and K_{t+1} .

4 CRRA Utility

Suppose

$$u^l(c, l) = \eta \ln c + (1 - \eta) \ln l \quad (33)$$

where $\eta \in [0, 1]$. Then

$$\mathcal{L}_{intra}^l = \eta \ln c + (1 - \eta) \ln l + \lambda[E - c - wl] + \mu[1 - l]$$

$$\frac{\partial \mathcal{L}_{intra}^l}{\partial c} = \frac{\eta}{c} - \lambda = 0$$

$$\frac{\partial \mathcal{L}_{intra}^l}{\partial l} = \frac{1 - \eta}{l} - \lambda w - \mu = 0$$

$$\frac{1 - \eta}{l} - \frac{\eta}{c} w \geq 0$$

with equality if $l < 1$.

Let us suppose first that $l < 1$. Then

$$c = \frac{\eta}{1 - \eta} wl$$

$$E = \frac{\eta}{1 - \eta} wl + wl = \frac{wl}{1 - \eta}$$

$$l = \frac{1 - \eta}{w} E$$

$$c = \eta E.$$

If $l = 1$, then

$$c = E - w$$

$$1 - \eta - \frac{\eta w}{E - w} > 0$$

$$1 - \eta > \frac{\eta w}{E - w}$$

$$E - w > \frac{\eta w}{1 - \eta}$$

$$E > \left[1 + \frac{\eta}{1 - \eta} \right] w = \frac{w}{1 - \eta}$$

Thus the policy functions are

$$c(E, w) = \begin{cases} \eta E & E < \frac{w}{1 - \eta} \\ E - w & E \geq \frac{w}{1 - \eta} \end{cases} \quad (34)$$

and

$$l(E, w) = \begin{cases} \frac{1 - \eta}{w} E & E < \frac{w}{1 - \eta} \\ 1 & E \geq \frac{w}{1 - \eta} \end{cases} \quad (35)$$

Then

$$v^l(E, w) = \begin{cases} \eta \ln(\eta E) + (1 - \eta) \ln\left(\frac{1-\eta}{w} E\right) & E < \frac{w}{1-\eta} \\ \eta \ln(E - w) & E \geq \frac{w}{1-\eta} \end{cases}$$

Thus the value function is

$$v^l(E, w) = \begin{cases} \eta \ln(\eta) + (1 - \eta) \ln\left(\frac{1-\eta}{w}\right) + \ln(E) & E < \frac{w}{1-\eta} \\ \eta \ln(E - w) & E \geq \frac{w}{1-\eta} \end{cases} \quad (36)$$

Then

$$\begin{aligned} \lim_{E \uparrow \frac{w}{1-\eta}} v^l(E, w) &= \eta \ln\left(\frac{\eta}{1-\eta} w\right) + (1 - \eta) \ln(1) = \eta \ln\left(\frac{\eta}{1-\eta} w\right) \\ \lim_{E \downarrow \frac{w}{1-\eta}} v^l(E, w) &= \eta \ln\left(\frac{w}{1-\eta} - w\right) = \eta \ln\left(\frac{\eta}{1-\eta} w\right), \end{aligned}$$

so v^l is continuous.

$$\frac{\partial v^l(E, w)}{\partial E} = \begin{cases} \frac{1}{E} & E < \frac{w}{1-\eta} \\ \frac{\eta}{E-w} & E \geq \frac{w}{1-\eta} \end{cases} \quad (37)$$

$$\lim_{E \uparrow \frac{w}{1-\eta}} \frac{\partial v^l(E, w)}{\partial E} = \frac{1-\eta}{w}$$

$$\lim_{E \downarrow \frac{w}{1-\eta}} \frac{\partial v^l(E, w)}{\partial E} = \frac{\eta}{\frac{w}{1-\eta} - w} = \frac{\eta}{\frac{1-(1-\eta)}{1-\eta} w} = \frac{1-\eta}{w}.$$

Thus v^l is differentiable with respect to E .

If we make the further assumption that $e_0 = 1$ and $e_1 = 0$ so households only work while young, things simplify more. Then

$$l(E_{t+1,1}^l, (1 - \tau_{t+1}^l)W_{t+1}e_1) = l(E_{t+1,1}^l, 0) = 1.$$

$$v^l(E_{t+1,1}^l, 0) = \eta \ln(E_{t+1,1}^l).$$

The lifetime budget constraint is

$$E_{t,0}^l + \frac{E_{t+1,1}^l}{1 + (1 - \tau_{t+1}^k)r_{t+1}} = (1 - \tau_t^l)W_t.$$

$$E_{t,0}^l \leq (1 - \tau_t^l)W_t < \frac{(1 - \tau_t^l)W_t}{1 - \eta}$$

since $0 \leq \eta \leq 1$. Thus

$$l(E_{t,0}^l, (1 - \tau_t^l)W_t) = \frac{1 - \eta}{(1 - \tau_t^l)W_t} E_{t,0}^l$$

$$v^l(E_{t,0}^l, (1 - \tau_t^l)W_t) = \eta \ln(\eta) + (1 - \eta) \ln\left(\frac{1 - \eta}{(1 - \tau_t^l)W_t}\right) + \ln(E_{t,0}^l)$$

The Lagrangian is

$$\begin{aligned} \mathcal{L}_{inter}^l &= \eta \ln(\eta) + (1 - \eta) \ln\left(\frac{1 - \eta}{(1 - \tau_t^l)W_t}\right) + \ln(E_{t,0}^l) + \beta^l \eta \ln(E_{t+1,1}^l) \\ &\quad + \lambda_t \left[(1 - \tau_t^l)W_t e_0 - E_{t,0}^l - \frac{E_{t+1,1}^l}{1 + (1 - \tau_{t+1}^k)r_{t+1}} \right]. \end{aligned}$$

The first-order conditions are

$$\begin{aligned} \frac{\partial \mathcal{L}_{inter}^l}{\partial E_{t,0}^l} &= \frac{1}{E_{t,0}^l} - \lambda_t = 0 \\ \frac{\partial \mathcal{L}_{inter}^l}{\partial E_{t+1,1}^l} &= \frac{\beta^l \eta}{E_{t+1,1}^l} - \frac{\lambda_t}{1 + (1 - \tau_{t+1}^k)r_{t+1}} = 0 \\ \frac{\beta^l \eta}{E_{t+1,1}^l} &= \frac{1}{1 + (1 - \tau_{t+1}^k)r_{t+1}} \frac{1}{E_{t,0}^l} \\ E_{t+1,1}^l &= \beta^l \eta [1 + (1 - \tau_{t+1}^k)r_{t+1}] E_{t,0}^l \\ E_{t,0}^l + \frac{\beta^l \eta [1 + (1 - \tau_{t+1}^k)r_{t+1}] E_{t,0}^l}{1 + (1 - \tau_{t+1}^k)r_{t+1}} &= (1 - \tau_t^l)W_t \\ (1 + \beta^l \eta)E_{t,0}^l &= (1 - \tau_t^l)W_t \\ E_{t,0}^l &= \frac{1 - \tau_t^l}{1 + \beta^l \eta} W_t \\ E_{t+1,1}^l &= \frac{\beta^l \eta}{1 + \beta^l \eta} [1 + (1 - \tau_{t+1}^k)r_{t+1}] (1 - \tau_t^l)W_t \\ l_{t,0} = l(E_{t,0}^l, (1 - \tau_t^l)W_t) &= \frac{1 - \eta}{(1 - \tau_t^l)W_t} E_{t,0}^l = \frac{1 - \eta}{1 + \beta^l \eta} < 1 - \eta \leq 1 \quad (38) \end{aligned}$$

Thus

$$N_t = \mu(1 - l_{t,0}) = \mu \left[1 - \frac{1 - \eta}{1 + \beta^l \eta} \right] = \mu \left[\frac{1 + \beta^l \eta - 1 + \eta}{1 + \beta^l \eta} \right] = \mu \frac{1 + \beta^l \eta}{1 + \beta^l \eta}$$

$$1 - l_{t,0} = \frac{1 + \beta^l \eta}{1 + \beta^l \eta} \eta = \frac{\eta + \beta^l \eta}{1 + \beta^l \eta} \leq 1.$$

$$\frac{\partial}{\partial \eta} (1 - l_{t,0}) = \frac{1 + \beta^l \eta}{1 + \beta^l \eta} - (1 + \beta^l \eta) \frac{\beta^l}{(1 + \beta^l \eta)^2} = (1 + \beta^l \eta) \frac{1 + \beta^l \eta - \beta^l \eta}{(1 + \beta^l \eta)^2} = \frac{1 + \beta^l \eta}{(1 + \beta^l \eta)^2} > 0$$

$$\frac{\partial}{\partial \beta^l} (1 - l_{t,0}) = \frac{\eta}{1 + \beta^l \eta} - \frac{(1 + \beta^l \eta) \eta^2}{(1 + \beta^l \eta)^2} = \eta \frac{1 + \beta^l \eta - (1 + \beta^l \eta) \eta}{(1 + \beta^l \eta)^2} = \eta \frac{1 - \eta}{(1 + \beta^l \eta)^2} \geq 0.$$

Thus $1 - l_{t,0}$ is maximized when $\eta = 1$ and $\beta^l \rightarrow \infty$, so $1 - l_{t,0} = 1$.

$$c'_{t,0} = c(E^l_{t,0}, (1 - \tau^l_t)W_t) = \eta E^l_{t,0} = \frac{\eta}{1 + \beta^l \eta} (1 - \tau^l_t)W_t \quad (39)$$

$$c'_{t+1,1} = c(E^l_{t+1,1}, 0) = E^l_{t+1,1} = \frac{\beta^l \eta}{1 + \beta^l \eta} [1 + (1 - \tau^k_{t+1})r_{t+1}] (1 - \tau^l_t)W_t \quad (40)$$

Most importantly,

$$k^l_{t+1} = (1 - \tau^l_t)W_t - E^l_{t,0} = (1 - \tau^l_t)W_t - \frac{1 - \tau^l_t}{1 + \beta^l \eta} W_t = \frac{\beta^l \eta}{1 + \beta^l \eta} (1 - \tau^l_t)W_t \quad (41)$$

Total lifetime income is

$$\frac{1 + \beta^l}{1 + \beta^l \eta} \eta (1 - \tau^l_t)W_t.$$

Total lifetime consumption is

$$\begin{aligned} c'_{t,0} + \frac{c'_{t+1,1}}{1 + (1 - \tau^k_{t+1})r_{t+1}} &= \frac{\eta}{1 + \beta^l \eta} (1 - \tau^l_t)W_t + \frac{\beta^l \eta}{1 + \beta^l \eta} (1 - \tau^l_t)W_t \\ &= \frac{(1 + \beta^l) \eta}{1 + \beta^l \eta} (1 - \tau^l_t)W_t \end{aligned}$$

This example clarifies the lessons of the model. A laborer's utility depends on the after-tax wage and the after-tax return on capital. But in the steady state, the after-tax return on capital is fixed by the preferences of capitalists, so ultimately the laborer's utility only depends on the after-tax wage. A decrease in capital taxes will increase the capital stock and increase the wage, but there will be some optimal combination of the two tax rates that maximizes the laborer's after-tax wage.

In this case, the capitalist's problem can be expressed in terms of a Bellman equation

$$v_t^c(k_t^c, K_t) = \max u^c(c_t^c) + \beta^c v_{t+1}(k_{t+1}^c, K_{t+1}) \quad (42)$$

subject to

$$\begin{aligned} c_t^c + k_{t+1}^c &= (1 + (1 - \tau^k_t)r_t)k_t^c \\ K_{t+1} &= \mu \frac{\beta^l \eta}{1 + \beta^l \eta} (1 - \tau^l_t)F_N \left(K_t, \mu \frac{1 + \beta^l}{1 + \beta^l \eta} \eta \right) + (1 - \mu)k_{t+1}^c \end{aligned} \quad (43)$$

$$r_t = F_K \left(K_t, \mu \frac{1 + \beta^l}{1 + \beta^l \eta} \eta \right) - \delta. \quad (44)$$

If we further specialize to the case of Cobb-Douglas utility, then (43) simplifies to

$$K_{t+1} = \mu \frac{\beta^l \eta}{1 + \beta^l \eta} (1 - \tau^l_t) (1 - \alpha) \left(\frac{K_t}{\mu \frac{1 + \beta^l}{1 + \beta^l \eta} \eta} \right)^\alpha + (1 - \mu)k_{t+1}^c$$

$$K_{t+1} = (1 - \tau_t^l) \frac{(1 - \alpha)\beta^l}{(1 + \beta^l)^\alpha} K_t^\alpha \left(\frac{\mu\eta}{1 + \beta^l\eta} \right)^{1-\alpha} + (1 - \mu)k_{t+1}^c \quad (45)$$

Likewise, (44) simplifies to

$$r_t = \alpha \left(\frac{1 + \beta^l\eta}{1 + \beta^l} \frac{K_t}{\mu\eta} \right)^{\alpha-1} - \delta. \quad (46)$$

The Lagrangian for the capitalist is

$$\begin{aligned} \mathcal{L}_t^c = & u^c \left(\left(1 + (1 - \tau_t^k) \left(\alpha \left(\frac{1 + \beta^l\eta}{1 + \beta^l} \frac{K_t}{\mu\eta} \right)^{\alpha-1} - \delta \right) \right) k_t^c - k_{t+1}^c \right) \\ & + \beta^c v_{t+1} \left(k_{t+1}^c, (1 - \tau_t^l) \frac{(1 - \alpha)\beta^l}{(1 + \beta^l)^\alpha} K_t^\alpha \left(\frac{\mu\eta}{1 + \beta^l\eta} \right)^{1-\alpha} + (1 - \mu)k_{t+1}^c \right). \end{aligned} \quad (47)$$

Thus

$$\begin{aligned} \frac{\partial \mathcal{L}_t^c}{\partial k_{t+1}^c} = & -(u^c)' \left(\left(1 + (1 - \tau_t^k) \left(\alpha \left(\frac{1 + \beta^l\eta}{1 + \beta^l} \frac{K_t}{\mu\eta} \right)^{\alpha-1} - \delta \right) \right) k_t^c - k_{t+1}^c \right) \\ & + \beta^c \frac{\partial v_{t+1}}{\partial k^c} \left(k_{t+1}^c, (1 - \tau_t^l) \frac{(1 - \alpha)\beta^l}{(1 + \beta^l)^\alpha} K_t^\alpha \left(\frac{\mu\eta}{1 + \beta^l\eta} \right)^{1-\alpha} + (1 - \mu)k_{t+1}^c \right) \\ & + (1 - \mu)\beta^c \frac{\partial v_{t+1}}{\partial K} \left(k_{t+1}^c, (1 - \tau_t^l) \frac{(1 - \alpha)\beta^l}{(1 + \beta^l)^\alpha} K_t^\alpha \left(\frac{\mu\eta}{1 + \beta^l\eta} \right)^{1-\alpha} + (1 - \mu)k_{t+1}^c \right) \end{aligned} \quad (48)$$

By the Envelope Theorem,

$$\frac{\partial v_t^c}{\partial k^c} = \frac{\partial \mathcal{L}_t^c}{\partial k^c} = (1 + (1 - \tau_t^k)r_t) (u^c)'(c_t^c).$$

Thus the Euler equation simplifies to

$$(u^c)'(c_t^c) = \beta^c (1 + (1 - \tau_{t+1}^k)r_{t+1}) (u^c)'(c_{t+1}^c) + (1 - \mu)\beta^c \frac{\partial v_{t+1}}{\partial K} (k_{t+1}^c, K_{t+1}). \quad (49)$$

Absent the last term, this is just the normal Euler equation. The new term reflects the effect of the capitalist's capital on the behavior of laborers.

$$\begin{aligned} \frac{\partial v_t^c}{\partial K} = & \frac{\partial \mathcal{L}_t^c}{\partial K} = (1 - \tau_t^k) \frac{\alpha(\alpha - 1)}{K_t} \left(\frac{1 + \beta^l\eta}{1 + \beta^l} \frac{K_t}{\mu\eta} \right)^{\alpha-1} (u^c)'(c_t^c) \\ & + \beta^c \frac{\partial v_{t+1}^c}{\partial K} (k_{t+1}^c, K_{t+1}) (1 - \tau_t^l) \frac{\alpha(1 - \alpha)\beta^l}{(1 + \beta^l)^\alpha} K_t^{\alpha-1} \left(\frac{\mu\eta}{1 + \beta^l\eta} \right)^{1-\alpha} \end{aligned} \quad (50)$$

The first term is the effect of having a higher capital stock next period, which is to lower the rate of return on capital. The second term is the effect on

laborer's wages of having more capital next period, which will be to save more and further raise the capital stock, which will again lower the rate of return on capital. Thus $\partial v_t^c / \partial K$ is negative. The capitalist will save less than would otherwise be optimal to increase the rate of return just as a monopolist restricts output to increase revenue. Just as is the case with optimal irrational behavior, obeying the normal Euler equation is not optimal for the capitalist. However, the capitalist wants a smaller capital stock whereas an optimal irrational social planner wants a bigger capital stock.

In the steady state, if $\partial v / \partial K < 0$, then

$$(u^c)'(c_*^c) = \beta^c(1 + (1 - \tau^k)r^*)(u^c)'(c_*^c) + (1 - \mu)\beta^c \frac{\partial v}{\partial K}(k_*^c, K^*)$$

$$[1 - \beta^c(1 + (1 - \tau^k)r^*)](u^c)'(c_*^c) = (1 - \mu)\beta^c \frac{\partial v}{\partial K}(k_*^c, K^*)$$

Thus we need

$$1 - \beta^c(1 + (1 - \tau^k)r^*) < 0$$

$$\frac{1}{\beta^c} < 1 + (1 - \tau^k)r^*$$

$$r^* > \frac{\frac{1}{\beta^c} - 1}{1 - \tau^k} = \frac{1 - \beta^c}{\beta^c(1 - \tau^k)},$$

where the latter would be the interest rate if the capitalist did not take into account the pecuniary externality. Thus the capital stock will be lower than it would be if the capitalist behaved competitively.

Suppose τ^k and τ^l are constant. Let (k_*^c, K^*) be the steady state such that $k_{t+1}^c = k_t^c = k_*^c$, $K_{t+1} = K_t = K^*$, and $v_t^c(k, K) = v^c(k, K)$. Then (50) simplifies to

$$\begin{aligned} \frac{\partial v^c}{\partial K}(k_*^c, K^*) &= (1 - \tau^k) \frac{\alpha(\alpha - 1)}{K^*} \left(\frac{1 + \beta^l \eta K^*}{1 + \beta^l \mu \eta} \right)^{\alpha - 1} (u^c)'(c_*^c) \\ &\quad + \beta^c \frac{\partial v^c}{\partial K}(k_*^c, K^*) (1 - \tau^l) \frac{\alpha(1 - \alpha)\beta^l}{(1 + \beta^l)^\alpha} \left(\frac{\mu \eta}{(1 + \beta^l \eta)K^*} \right)^{1 - \alpha} \end{aligned} \quad (51)$$

Solving for the partial derivative,

$$\frac{\partial v^c}{\partial K}(k_*^c, K^*) = \frac{(1 - \tau^k) \frac{\alpha(\alpha - 1)}{K^*} \left(\frac{1 + \beta^l \eta K^*}{1 + \beta^l \mu \eta} \right)^{\alpha - 1} (u^c)'(c_*^c)}{1 - \beta^c(1 - \tau^l) \frac{\alpha(1 - \alpha)\beta^l}{(1 + \beta^l)^\alpha} \left(\frac{\mu \eta}{(1 + \beta^l \eta)K^*} \right)^{1 - \alpha}}. \quad (52)$$

This only makes sense if

$$\beta^c(1 - \tau^l) \frac{\alpha(1 - \alpha)\beta^l}{(1 + \beta^l)^\alpha} \left(\frac{\mu \eta}{(1 + \beta^l \eta)K^*} \right)^{1 - \alpha} < 1. \quad (53)$$

Note that $\alpha, \eta \in [0, 1]$ and $\beta^c, \tau^l \in [0, 1)$, though we only have $\beta^l > 0$. However, we do have

$$\begin{aligned} \frac{1 + \beta^l}{1 + \beta^l \eta} \eta &\leq 1 \\ \frac{\eta}{1 + \beta^l \eta} &\leq \frac{1}{1 + \beta^l} \\ \frac{\beta^l}{(1 + \beta^l)^\alpha} \left(\frac{\eta}{1 + \beta^l \eta} \right)^{1-\alpha} &\leq \frac{\beta^l}{1 + \beta^l} \leq 1. \\ r^* + \delta &= \alpha \left(\frac{1 + \beta^l \eta K^*}{1 + \beta^l \mu \eta} \right)^{\alpha-1} = \alpha \left(\frac{1 + \beta^l \mu \eta}{1 + \beta^l \eta K^*} \right)^{1-\alpha} \end{aligned}$$

Then the condition (53) can be rewritten

$$\beta^c (1 - \tau^l) (1 - \alpha) \frac{\beta^l}{1 + \beta^l} (r^* + \delta) < 1.$$

I was endeavoring to show that this condition always holds, but that is probably not true. One can imagine that a small increase in the capital stock would have a divergent effect on the capitalist's utility, particularly if $\beta^l > \beta^c$. So there are probably parameters where there is no positive steady state capital stock. It may be optimal for the capitalists to drive the economy to extinction, ensuring that they keep earning a high return to capital.

In order for a steady state to exist, the resulting K^* must be high enough so a small change in capital does not have a big effect on the rate of return on capital.

4.1 Terminal Value Function

Suppose that at time $T + 1$ the capitalist dies and there will only be a laborer. Thus

$$v_T^c(k_T^c, K_T) = \max u^c(c_T^c)$$

subject to

$$\begin{aligned} c_T^c &= (1 + (1 - \tau_T^k) r_T) k_T^c \\ r_T &= F_K \left(K_T, \mu \frac{1 + \beta^l}{1 + \beta^l \eta} \eta \right) - \delta. \end{aligned}$$

Thus the terminal value function is

$$v_T^c(k_T^c, K_T) = u \left(\left(1 + (1 - \tau_T^k) \left[F_K \left(K_T, \mu \frac{1 + \beta^l}{1 + \beta^l \eta} \eta \right) - \delta \right] \right) k_T^c \right) \quad (54)$$

$$\begin{aligned}
\frac{\partial v_T^c(k_T^c, K_T)}{\partial k_T^c} &= \left(1 + (1 - \tau_T^k) \left[F_K \left(K_T, \mu \frac{1 + \beta^l}{1 + \beta^l \eta} \right) - \delta \right] \right) \\
&\quad \times u' \left(\left(1 + (1 - \tau_T^k) \left[F_K \left(K_T, \mu \frac{1 + \beta^l}{1 + \beta^l \eta} \right) - \delta \right] \right) k_T^c \right) \\
&\geq 0
\end{aligned} \tag{55}$$

since $\delta \in [0, 1]$ and $\tau_t^k \in [0, 1]$.

$$\begin{aligned}
\frac{\partial v_T^c(k_T^c, K_T)}{\partial K_T} &= (1 - \tau_T^k) F_{KK} \left(K_T, \mu \frac{1 + \beta^l}{1 + \beta^l \eta} \right) \\
&\quad \times u' \left(\left(1 + (1 - \tau_T^k) \left[F_K \left(K_T, \mu \frac{1 + \beta^l}{1 + \beta^l \eta} \right) - \delta \right] \right) k_T^c \right) \\
&< 0.
\end{aligned} \tag{56}$$

Let us assume that $\frac{\partial v_{t+1}^c}{\partial k_{t+1}^c} > 0$ and $\frac{\partial v_{t+1}^c}{\partial K_{t+1}} < 0$. Then by the Envelope Theorem,

$$\frac{\partial v_t^c(k_t^c, K_t)}{\partial k_t^c} = \frac{\partial \mathcal{L}_t^c(k_t^c, K_t, \lambda_t)}{\partial k_t^c}$$

and

$$\frac{\partial v_t^c(k_t^c, K_t)}{\partial K_t} = \frac{\partial \mathcal{L}_t^c(k_t^c, K_t, \lambda_t)}{\partial K_t},$$

where

$$\begin{aligned}
\mathcal{L}_t^c(k_t^c, K_t, \lambda_t) &= u^c(c_t^c) + \beta^c v_{t+1} \left(k_{t+1}^c, \mu \frac{\beta^l \eta}{1 + \beta^l \eta} (1 - \tau_t^l) F_N \left(K_t, \mu \frac{1 + \beta^l}{1 + \beta^l \eta} \right) + (1 - \mu) k_{t+1}^c \right) \\
&\quad + \lambda_t \left[\left(1 + (1 - \tau_t^k) \left[F_K \left(K_t, \mu \frac{1 + \beta^l}{1 + \beta^l \eta} \right) - \delta \right] \right) k_t^c - c_t^c - k_{t+1}^c \right]
\end{aligned}$$

Note that

$$\lambda_t = u'(c_t^c).$$

$$\frac{\partial v_t^c(k_t^c, K_t)}{\partial k_t^c} = \frac{\partial \mathcal{L}_t^c(k_t^c, K_t, \lambda_t)}{\partial k_t^c} = \left(1 + (1 - \tau_t^k) \left[F_K \left(K_t, \mu \frac{1 + \beta^l}{1 + \beta^l \eta} \right) - \delta \right] \right) u'(c_t^c) > 0$$

$$\begin{aligned}
\frac{\partial \mathcal{L}_t^c(k_t^c, K_t, \lambda_t)}{\partial K_t} &= \beta^c \frac{\partial v_{t+1}}{\partial K_{t+1}}(k_{t+1}^c, K_{t+1}) \mu \frac{\beta^l \eta}{1 + \beta^l \eta} (1 - \tau_t^l) F_{KN} \left(K_t, \mu \frac{1 + \beta^l}{1 + \beta^l \eta} \right) \\
&\quad + u'(c_t^c) (1 - \tau_t^k) F_{KK} \left(K_t, \mu \frac{1 + \beta^l}{1 + \beta^l \eta} \right) k_t^c
\end{aligned}$$

Suppose we have a production function that satisfies

$$F(K, N) = f \left(\frac{K}{N} \right) N$$

where $f'(k) > 0$ and $f''(k) < 0$. Then

$$F_K(K, N) = f' \left(\frac{K}{N} \right)$$

$$F_{KN}(K, N) = f'' \left(\frac{K}{N} \right) \left(-\frac{K}{N^2} \right) = \frac{\partial w(K, N)}{\partial K} > 0$$

Thus $\frac{\partial v_t^c(k_t^c, K_t)}{\partial K_t} < 0$.

We can also write the objective function as

$$\begin{aligned} M(k_{t+1}^c) &= u^c \left(\left(1 + (1 - \tau_t^k) \left[F_K \left(K_t, \mu \frac{1 + \beta^l}{1 + \beta^l \eta} \right) - \delta \right] \right) k_t^c - k_{t+1}^c \right) \\ &\quad + \beta^c v_{t+1} \left(k_{t+1}^c, \mu \frac{\beta^l \eta}{1 + \beta^l \eta} (1 - \tau_t^l) F_N \left(K_t, \mu \frac{1 + \beta^l}{1 + \beta^l \eta} \right) + (1 - \mu) k_{t+1}^c \right) \end{aligned}$$

$$\begin{aligned} M'(k_{t+1}^c) &= -(u^c)' \left(\left(1 + (1 - \tau_t^k) \left[F_K \left(K_t, \mu \frac{1 + \beta^l}{1 + \beta^l \eta} \right) - \delta \right] \right) k_t^c - k_{t+1}^c \right) \\ &\quad + \beta^c \frac{\partial v_{t+1}}{\partial k} \left(k_{t+1}^c, \mu \frac{\beta^l \eta}{1 + \beta^l \eta} (1 - \tau_t^l) F_N \left(K_t, \mu \frac{1 + \beta^l}{1 + \beta^l \eta} \right) + (1 - \mu) k_{t+1}^c \right) \\ &\quad + \beta^c (1 - \mu) \frac{\partial v_{t+1}}{\partial K} \left(k_{t+1}^c, \mu \frac{\beta^l \eta}{1 + \beta^l \eta} (1 - \tau_t^l) F_N \left(K_t, \mu \frac{1 + \beta^l}{1 + \beta^l \eta} \right) + (1 - \mu) k_{t+1}^c \right) \end{aligned}$$

$$\begin{aligned} M''(k_{t+1}^c) &= (u^c)'' \left(\left(1 + (1 - \tau_t^k) \left[F_K \left(K_t, \mu \frac{1 + \beta^l}{1 + \beta^l \eta} \right) - \delta \right] \right) k_t^c - k_{t+1}^c \right) \\ &\quad + \beta^c \frac{\partial^2 v_{t+1}}{\partial k^2} \left(k_{t+1}^c, \mu \frac{\beta^l \eta}{1 + \beta^l \eta} (1 - \tau_t^l) F_N \left(K_t, \mu \frac{1 + \beta^l}{1 + \beta^l \eta} \right) + (1 - \mu) k_{t+1}^c \right) \\ &\quad + 2\beta^c (1 - \mu) \frac{\partial^2 v_{t+1}}{\partial k \partial K} \left(k_{t+1}^c, \mu \frac{\beta^l \eta}{1 + \beta^l \eta} (1 - \tau_t^l) F_N \left(K_t, \mu \frac{1 + \beta^l}{1 + \beta^l \eta} \right) + (1 - \mu) k_{t+1}^c \right) \\ &\quad + \beta^c (1 - \mu)^2 \frac{\partial^2 v_{t+1}}{\partial K^2} \left(k_{t+1}^c, \mu \frac{\beta^l \eta}{1 + \beta^l \eta} (1 - \tau_t^l) F_N \left(K_t, \mu \frac{1 + \beta^l}{1 + \beta^l \eta} \right) + (1 - \mu) k_{t+1}^c \right) \end{aligned}$$

The first term is negative. The second term is presumably negative. The remaining terms are of indeterminate size.

However, u^c will satisfy an Inada condition at 0 and $v_{t+1}(k, K)$ will also satisfy an Inada condition at $k = 0$. Thus there must be a global maximum.

Suppose F is Cobb-Douglas with share of capital α and u^c is log. Let $\tau^k = \delta = N = 1$. Then

$$v_T(k, K) = \ln [\alpha K^{\alpha-1} k]$$

$$\begin{aligned}
\frac{\partial v_T}{\partial k} &= \frac{1}{k} > 0 \\
\frac{\partial v_T}{\partial K} &= \frac{\alpha - 1}{K} < 0 \\
\frac{\partial^2 v_T}{\partial k^2} &= -\frac{1}{k^2} < 0 \\
\frac{\partial^2 v_T}{\partial K^2} &= -\frac{(\alpha - 1)}{K^2} = \frac{1 - \alpha}{K^2} > 0
\end{aligned}$$

Thus the value function is definitely not concave.

While we cannot solve analytically for $v_t(k, K)$ for all k , we can approximate $v_t(k, K)$ in the limit of small k . However, the following approximation is only valid for $s > 0$. We will have to treat the case of $s = 0$ separately.

$$v_{T-1}^c(k_{T-1}^c, K_{T-1}) = \max u^c(c_{T-1}^c) + \beta^c v_T(k_{T-1}^c, K_{T-1}) \quad (57)$$

subject to

$$\begin{aligned}
c_{T-1}^c + k_T^c &= (1 + (1 - \tau_{T-1}^k)r_{T-1})k_{T-1}^c \\
K_T &= \mu \frac{\beta^l \eta}{1 + \beta^l \eta} (1 - \tau_{T-1}^l) F_N \left(K_{T-1}, \mu \frac{1 + \beta^l}{1 + \beta^l \eta} \eta \right) + (1 - \mu)k_T^c \\
r_{T-1} &= F_K \left(K_{T-1}, \mu \frac{1 + \beta^l}{1 + \beta^l \eta} \eta \right) - \delta.
\end{aligned}$$

Let

$$N = \mu \frac{1 + \beta^l}{1 + \beta^l \eta} \eta \quad (58)$$

and

$$s = \frac{\beta^l \eta}{1 + \beta^l \eta}. \quad (59)$$

In the limit of small k_{T-1} ,

$$K_T = (1 - \alpha)\mu s (1 - \tau_{T-1}^l) \left(\frac{K_{T-1}}{N} \right)^\alpha. \quad (60)$$

First consider the case where $\gamma = 1$.

$$v_T(k_T, K_T) = \ln \left[\left((1 + (1 - \tau_T^k)r(K_T)) \right) k_t \right]$$

The capitalist's problem in this limit is

$$\max \ln(c_{T-1}^c) + \beta^c \ln \left((1 + (1 - \tau_T^k)r(K_T)) k_T^c \right)$$

subject to

$$\begin{aligned}
c_{T-1}^c + k_T^c &= (1 + (1 - \tau_{T-1}^k)r_{T-1})k_{T-1}^c \\
L_{T-1}^c &= \ln(c_{T-1}^c) + \beta^c \ln \left((1 + (1 - \tau_{T-1}^k)r_{T-1})k_{T-1}^c - c_{T-1}^c \right)
\end{aligned}$$

$$\begin{aligned}\frac{\partial L_{T-1}^c}{\partial c_{T-1}^c} &= \frac{1}{c_{T-1}^c} - \frac{\beta^c}{(1 + (1 - \tau_{T-1}^k)r_{T-1})k_{T-1}^c - c_{T-1}^c} = 0 \\ (1 + (1 - \tau_{T-1}^k)r_{T-1})k_{T-1}^c - c_{T-1}^c &= \beta^c c_{T-1} \\ c_{T-1}^c(k_{T-1}^c, K_{T-1}) &= \frac{(1 + (1 - \tau_{T-1}^k)r(K_{T-1}))k_{T-1}^c}{1 + \beta^c}\end{aligned}\quad (61)$$

$$\begin{aligned}k_T^c(k_{T-1}^c, K_{T-1}) &= (1 + (1 - \tau_{T-1}^k)r_{T-1}(K_{T-1}))k_{T-1}^c - \frac{(1 + (1 - \tau_{T-1}^k)r_{T-1}(K_{T-1}))k_{T-1}^c}{1 + \beta^c} \\ k_T^c(k_{T-1}^c, K_{T-1}) &= \frac{\beta^c}{1 + \beta^c}(1 + (1 - \tau_{T-1}^k)r_{T-1}(K_{T-1}))k_{T-1}^c\end{aligned}\quad (62)$$

Let

$$R_t(K_t) = 1 + (1 - \tau_t^k)r(K_t)\quad (63)$$

while for $s > t$,

$$R_s(K_t) = R_s(K_s(K_t))$$

since in the small k_t limit, K_s is simply a function of K_t .

$$\begin{aligned}v_{T-1}(k_{T-1}^c, K_{T-1}) &= \ln\left(\frac{R_{T-1}(K_{T-1})k_{T-1}^c}{1 + \beta^c}\right) + \beta^c \ln\left[R_T(K_{T-1})\frac{\beta^c}{1 + \beta^c}R_{T-1}(K_{T-1})k_{T-1}^c\right] \\ v_{T-1}(k_{T-1}^c, K_{T-1}) &= (1 + \beta^c) \ln(k_{T-1}^c) + \beta^c \ln[\beta^c R_T(K_{T-1})] + (1 + \beta^c) \ln\left(\frac{R_{T-1}(K_{T-1})}{1 + \beta^c}\right)\end{aligned}\quad (64)$$

$$v_T(k_T^c, K_T) = \ln(R_T(K_T)k_T^c).$$

Let us assume that

$$v_t(k_t^c, K_t) = M_t \ln(k_t^c) + D_t(K_t).\quad (65)$$

The capitalist's problem at t has Lagrangian

$$L_t(c_t^c, K_t) = \ln(c_t^c) + \beta^c [M_{t+1} \ln(R_t(K_t)k_t - c_t^c) + D_t(K_t)]\quad (66)$$

$$\begin{aligned}\frac{\partial L_t(c_t^c, K_t)}{\partial c_t^c} &= \frac{1}{c_t^c} - \frac{\beta^c M_{t+1}}{R_t(K_t)k_t - c_t^c} = 0 \\ R_t(K_t)k_t - c_t^c &= \beta^c M_{t+1}c_t^c\end{aligned}$$

$$c_t^c(k_t, K_t) = \frac{R_t(K_t)k_t}{1 + \beta^c M_{t+1}}\quad (67)$$

$$k_{t+1}^c(k_t, K_t) = \frac{\beta^c M_{t+1}}{1 + \beta^c M_{t+1}}R_t(K_t)k_t\quad (68)$$

$$v_t(k_t, K_t) = \ln\left(\frac{R_t(K_t)k_t}{1 + \beta^c M_{t+1}}\right) + \beta^c M_{t+1} \ln\left(\frac{\beta^c M_{t+1}}{1 + \beta^c M_{t+1}}R_t(K_t)k_t\right) + \beta^c D_{t+1}(K_{t+1}(K_t))$$

$$\begin{aligned}
v_t(k_t, K_t) &= (1 + \beta^c M_{t+1}) \ln(k_t) \\
&\quad + \ln\left(\frac{R_t(K_t)}{1 + \beta^c M_{t+1}}\right) + \beta^c M_{t+1} \ln\left(\frac{\beta^c M_{t+1}}{1 + \beta^c M_{t+1}} R_t(K_t)\right) + \beta^c D_{t+1}(K_{t+1}(K_t)) \\
M_t &= 1 + \beta^c M_{t+1} \\
M_t &= \frac{1 - (\beta^c)^{T-t+1}}{1 - \beta^c} \\
M_T &= \frac{1 - \beta^c}{1 - \beta^c} = 1 \\
M_{t+1} &= \frac{1 - (\beta^c)^{T-t}}{1 - \beta^c} \\
M_t &= 1 + \beta^c \frac{1 - (\beta^c)^{T-t}}{1 - \beta^c} = \frac{1 - \beta^c + \beta^c - (\beta^c)^{T-t+1}}{1 - \beta^c} = \frac{1 - (\beta^c)^{T-t+1}}{1 - \beta^c} \\
D_t(K_t) &= \ln\left(\frac{R_t(K_t)}{1 + \beta^c M_{t+1}}\right) + \beta^c M_{t+1} \ln\left(\frac{\beta^c M_{t+1}}{1 + \beta^c M_{t+1}} R_t(K_t)\right) + \beta^c D_{t+1}(K_{t+1}(K_t)) \\
&\hspace{15em} (69)
\end{aligned}$$

At the opposite extreme, in the limit of very large k_{T-1} ,

$$\begin{aligned}
K_{t+1} &= (1 - \mu)k_{t+1}^c. \\
v_T(k_T^c, K_T) &= \ln\left[\left(1 + (1 - \tau_T^k) \left[\alpha \left(\frac{K_T}{N}\right)^{\alpha-1} - \delta\right]\right) k_T^c\right] \\
v_T(k_T^c, K_T) &= \ln\left[\left(1 + (1 - \tau_T^k) \left[\alpha \left(\frac{1 - \mu}{N} k_{t+1}^c\right)^{\alpha-1} - \delta\right]\right) k_T^c\right] \\
v_T(k_T^c, K_T) &\approx \ln[(1 - \delta(1 - \tau_T^k))k_T^c] \\
L_{T-1}(c_{T-1}^c) &= \ln(c_{T-1}^c) + \beta^c \ln[(1 - \delta(1 - \tau_T^k))(R_{T-1}(K_{T-1})k_{T-1}^c - c_{T-1}^c)] \\
L'_{T-1}(c_{T-1}^c) &= \frac{1}{c_{T-1}^c} - \frac{\beta^c}{R_{T-1}(K_{T-1})k_{T-1}^c - c_{T-1}^c} = 0 \\
c_{T-1}^c &= \frac{R_{T-1}(K_{T-1})k_{T-1}^c}{1 + \beta^c} \\
K_T &= (1 - \mu) \frac{\beta^c}{1 + \beta^c} R_{T-1}(K_{T-1})k_{T-1}^c.
\end{aligned}$$

In this limit,

$$K_T = \frac{\beta^c}{1 + \beta^c} [1 - \delta(1 - \tau_{T-1}^k)] K_{T-1} < K_T.$$

For $\gamma \neq 1$, in the small k_t limit, suppose that

$$v_t(k_t^c, K_t) = M_t(K_t)u^c(k_t^c) + D(K_t). \quad (70)$$

We have that

$$v_T(k_T^c, K_T) = u^c(R_T(K_T)k_T^c) = R_T^{1-\gamma_c}(K_T)u(k_T^c), \quad (71)$$

so

$$M_T(K_T) = R_T^{1-\gamma_c}(K_T) \quad (72)$$

and

$$D_T(K_T) = 0. \quad (73)$$

Now suppose the ansatz is correct for $t + 1$. Then

$$L_t(c_t^c) = u^c(c_t^c) + \beta^c M_{t+1}(K_{t+1}(K_t))u^c(R_t(K_t)k_t^c - c_t^c) + \beta^c D_{t+1}(K_{t+1}(K_t)) \quad (74)$$

$$L'_t(c_t) = (c_t^c)^{-\gamma_c} - \beta^c M_{t+1}(K_{t+1}(K_t))(R_t(K_t)k_t^c - c_t^c)^{-\gamma_c} = 0$$

$$c_t^c = (\beta^c M_{t+1}(K_{t+1}(K_t)))^{-1/\gamma_c} (R_t(K_t)k_t^c - c_t^c)$$

$$\left[1 + (\beta^c M_{t+1}(K_{t+1}(K_t)))^{-1/\gamma_c}\right] c_t^c = (\beta^c M_{t+1}(K_{t+1}(K_t)))^{-1/\gamma_c} R_t(K_t)k_t^c$$

$$c_t^c(k_t^c, K_t) = \frac{(\beta^c M_{t+1}(K_{t+1}(K_t)))^{-1/\gamma_c}}{1 + (\beta^c M_{t+1}(K_{t+1}(K_t)))^{-1/\gamma_c}} R_t(K_t)k_t^c. \quad (75)$$

$$k_{t+1}^c(k_t^c, K_t) = R_t(K_t)k_t^c - \frac{(\beta^c M_{t+1}(K_{t+1}(K_t)))^{-1/\gamma_c}}{1 + (\beta^c M_{t+1}(K_{t+1}(K_t)))^{-1/\gamma_c}} R_t(K_t)k_t^c$$

$$k_{t+1}^c(k_t^c, K_t) = \frac{1}{1 + (\beta^c M_{t+1}(K_{t+1}(K_t)))^{-1/\gamma_c}} R_t(K_t)k_t^c \quad (76)$$

$$\begin{aligned} v_t(k_t^c, K_t) &= u^c \left(\frac{(\beta^c M_{t+1}(K_{t+1}(K_t)))^{-1/\gamma_c}}{1 + (\beta^c M_{t+1}(K_{t+1}(K_t)))^{-1/\gamma_c}} R_t(K_t)k_t^c \right) \\ &\quad + \beta^c M_{t+1}(K_{t+1}(K_t))u^c \left(\frac{1}{1 + (\beta^c M_{t+1}(K_{t+1}(K_t)))^{-1/\gamma_c}} R_t(K_t)k_t^c \right) \\ &\quad + \beta^c D_{t+1}(K_{t+1}(K_t)) \end{aligned}$$

$$\begin{aligned} v_t(k_t^c, K_t) &= \left[(\beta^c M_{t+1}(K_{t+1}(K_t)))^{\frac{\gamma_c-1}{\gamma_c}} + \beta^c M_{t+1}(K_{t+1}(K_t)) \right] \\ &\quad \times \left(\frac{R_t(K_t)}{1 + (\beta^c M_{t+1}(K_{t+1}(K_t)))^{-1/\gamma_c}} \right)^{1-\gamma_c} u^c(k_t^c) \\ &\quad + \beta^c D_{t+1}(K_{t+1}(K_t)) \end{aligned}$$

Thus

$$D_t(K_t) = \beta^c D_{t+1}(K_{t+1}(K_t)).$$

Since $D_T(K_T) = 0$, $D_t(K_t) = 0$.

$$M_t(K_t) = \left[(\beta^c M_{t+1}(K_{t+1}(K_t)))^{\frac{\gamma_c-1}{\gamma_c}} + \beta^c M_{t+1}(K_{t+1}(K_t)) \right] \left(\frac{R_t(K_t)}{1 + (\beta^c M_{t+1}(K_{t+1}(K_t)))^{-1/\gamma_c}} \right)^{1-\gamma_c}$$

$$\begin{aligned}
M_t(K_t) &= \beta^c M_{t+1}(K_{t+1}(K_t)) \left((\beta^c M_{t+1}(K_{t+1}(K_t)))^{-\frac{1}{\gamma_c}} + 1 \right) \left(\frac{R_t(K_t)}{1 + (\beta^c M_{t+1}(K_{t+1}(K_t)))^{-1/\gamma_c}} \right)^{1-\gamma_c} \\
M_t(K_t) &= \beta^c M_{t+1}(K_{t+1}(K_t)) R_t^{1-\gamma_c}(K_t) \left((\beta^c M_{t+1}(K_{t+1}(K_t)))^{-\frac{1}{\gamma_c}} + 1 \right)^{\gamma_c} \\
M_t(K_t) &= R_t^{1-\gamma_c}(K_t) \left((\beta^c M_{t+1}(K_{t+1}(K_t)))^{\frac{1}{\gamma_c}} + 1 \right)^{\gamma_c} \quad (77)
\end{aligned}$$

$$\begin{aligned}
c_t^c(k_t^c, K_t) &= \frac{(\beta^c M_{t+1}(K_{t+1}(K_t)))^{-1/\gamma_c}}{1 + (\beta^c M_{t+1}(K_{t+1}(K_t)))^{-1/\gamma_c}} R_t(K_t) k_t^c \\
&= \frac{R_t(K_t) k_t^c}{1 + (\beta^c M_{t+1}(K_{t+1}(K_t)))^{1/\gamma_c}}
\end{aligned}$$

In the special case where $K_t = K^*$ and R is constant,

$$M_t = R^{1-\gamma_c} \left((\beta_c M_{t+1})^{\frac{1}{\gamma_c}} + 1 \right)^{\gamma_c}$$

Let

$$m_t = \frac{M_t}{R^{1-\gamma_c}} = \left((\beta_c R^{1-\gamma_c} m_{t+1})^{\frac{1}{\gamma_c}} + 1 \right)^{\gamma_c}$$

and

$$\phi_c = (\beta_c R^{1-\gamma_c})^{-\frac{1}{\gamma_c}}$$

$$m_t^{\frac{1}{\gamma_c}} = 1 + \frac{m_{t+1}^{\frac{1}{\gamma_c}}}{\phi_c}$$

$$m_T = 1$$

$$m_t^{\frac{1}{\gamma_c}} = \frac{1 - \phi_c^{-(T-t+1)}}{1 - \phi_c^{-1}}$$

$$m_T^{1/\gamma_c} = 1$$

$$\begin{aligned}
m_t^{1/\gamma_c} &= 1 + \phi_c^{-1} \frac{1 - \phi_c^{-(T-(t+1)+1)}}{1 - \phi_c^{-1}} \\
&= 1 + \phi_c^{-1} \frac{1 - \phi_c^{-(T-t)}}{1 - \phi_c^{-1}} \\
&= \frac{1 - \phi_c^{-1} + \phi_c^{-1} - \phi_c^{-(T-t+1)}}{1 - \phi_c^{-1}} \\
&= \frac{1 - \phi_c^{-(T-t+1)}}{1 - \phi_c^{-1}} \\
m_t &= \left(\frac{1 - \phi_c^{-(T-t+1)}}{1 - \phi_c^{-1}} \right)^{\gamma_c}
\end{aligned}$$

$$\begin{aligned}
M_t &= \left(\frac{1 - \phi_c^{-(T-t+1)}}{1 - \phi_c^{-1}} \right)^{\gamma_c} R^{1-\gamma_c} \\
c_t(k_t) &= \frac{(\beta^c M_{t+1})^{-1/\gamma_c}}{1 + (\beta^c M_{t+1})^{-1/\gamma_c}} R k_t^c \\
(\beta^c M_{t+1})^{-1/\gamma_c} &= (\beta^c R^{1-\gamma_c} m_{t+1})^{-1/\gamma_c} = \phi_c \frac{1 - \phi_c^{-1}}{1 - \phi_c^{-(T-t)}} = \frac{\phi_c - 1}{1 - \phi_c^{-(T-t)}} \\
\frac{(\beta^c M_{t+1})^{-1/\gamma_c}}{1 + (\beta^c M_{t+1})^{-1/\gamma_c}} &= \frac{\frac{\phi_c - 1}{1 - \phi_c^{-(T-t)}}}{1 + \frac{\phi_c - 1}{1 - \phi_c^{-(T-t)}}} = \frac{\phi_c - 1}{1 - \phi_c^{-(T-t)} + \phi_c - 1} = \frac{1 - \phi_c^{-1}}{1 - \phi_c^{-(T-t+1)}} \\
c_t(k_t) &= \frac{1 - \phi_c^{-1}}{1 - \phi_c^{-(T-t+1)}} R k_t^c
\end{aligned}$$

Conversely, in the limit of large K , $R(K) \rightarrow 1 - \delta$, and so value and consumption functions should not depend on K . However, K has to get very, very, very large before we can make that approximation. It would not be practical to consider so large a state space since K will never get so large. So instead, we need to find k_{\max} and K_{\max} such that

$$K_{t+1}(k_t, K_{\max}) \leq K_{\max}$$

and

$$k_{t+1}(k_{\max}, K_t) \leq k_{\max}$$

for all $k_t \leq k_{\max}$ and $K_t \leq K_{\max}$. For the first, we have

$$K_{t+1}(k_t, K_t) = \mu s^l (1 - \tau_t^l) w(K_t) + (1 - \mu) k_{t+1}(k_t, K_t).$$

Note that

$$w(K) = (1 - \alpha) \left(\frac{K}{N} \right)^\alpha$$

is strictly concave. For

$$K^{1-\alpha} = \mu s^l \frac{1 - \alpha}{N^\alpha}$$

Thus for

$$\begin{aligned}
\bar{K} &= \left(\frac{\mu s^l (1 - \alpha)}{N^\alpha} \right)^{\frac{1}{1-\alpha}} \\
w'(K) &= \alpha (1 - \alpha) \frac{K^{\alpha-1}}{N^\alpha},
\end{aligned}$$

which is a decreasing function.

$$w'(\bar{K}) = \frac{\alpha(1 - \alpha)}{N^\alpha} \frac{N^\alpha}{\mu s^l (1 - \alpha)} = \frac{\alpha}{\mu s^l}$$

Thus for $K_t \geq \bar{K}$,

$$\frac{d}{dK_t} (\mu s^l (1 - \tau_t^l) w(K_t)) \leq \mu s^l (1 - \tau_t^l) \frac{\alpha}{\mu s^l} < 1$$

So

$$\mu s^l (1 - \tau_t^l) w(K_t) \leq K_t$$

for $K_t \geq \bar{K}$.

5 Myopic Laborers

Suppose that the laborers have $\beta^l = 0$, so they do not save. They just supply N elastically and get w without saving. If the capitalists are price-takers, they would then solve

$$\max v(K) = u_c(c_c) + \beta^c v(K')$$

subject to

$$c_c + K' = RK$$

where in equilibrium

$$R = R(K) = F_K(K, N) + 1 - \delta.$$

Then the Lagrangian is

$$L = u_c(c_c) + \beta^c v(K') + \lambda [RK - c_c - K']$$

$$\frac{\partial L}{\partial c_c} = (u_c)'(c_c) - \lambda = 0$$

$$\frac{\partial L}{\partial K'} = \beta^c v'(K') - \lambda = 0$$

$$v'(K) = \frac{\partial L}{\partial K} = \lambda R$$

$$(u_c)'(c_c) = \beta^c v'(K') = \beta^c R(K')(u_c)'(c_c) = \beta^c (F_K(K', N) + 1 - \delta)(u_c)'(c_c).$$

In the steady state, K^* solves

$$F_K(K^*, N) = \frac{1}{\beta^c} + \delta - 1. \quad (78)$$

If the capitalists are not price-takers and maximize RK instead (which would result if wages are still determined competitively), the problem becomes

$$\max v(K) = u_c(c_c) + \beta^c v(K')$$

subject to

$$c_c + K' = [F_K(K, N) + 1 - \delta]K.$$

$$L = u_c(c_c) + \beta^c v(K') + \lambda \{ [F_K(K, N) + 1 - \delta]K - c_c - K' \}$$

$$\frac{\partial L}{\partial c_c} = u'_c(c_c) - \lambda = 0$$

$$\frac{\partial L}{\partial K'} = \beta^c v'(K') - \lambda = 0$$

$$v'(K) = \frac{\partial L}{\partial K} = \lambda [F_K(K, N) + 1 - \delta + F_{KK}(K, N)K].$$

Thus, even if the laborers do not contribute to saving, it will matter if the capitalists know that the rate of return depends on K . Thus we will get

$$(u_c)'(c_c) = \beta^c v'(K') = \beta^c (F_K(K', N) + 1 - \delta + F_{KK}(K', N)K')(u_c)'(c'_c).$$

In the steady state,

$$F_K(K^{**}, N) + F_{KK}(K^{**}, N)K^{**} = \frac{1}{\beta^c} + \delta - 1.$$

For Cobb-Douglas, $F(K, N) = K^\alpha N^{1-\alpha}$,

$$F_K(K, N) = \alpha \left(\frac{K}{N}\right)^{\alpha-1}$$

$$F_{KK}(K, N) = \alpha(\alpha - 1)K^{\alpha-2}N^{1-\alpha}$$

$$KF_{KK}(K, N) = \alpha(\alpha - 1) \left(\frac{K}{N}\right)^{\alpha-1}$$

$$F_K(K, N) + KF_{KK}(K, N) = \alpha^2 \left(\frac{K}{N}\right)^{\alpha-1}$$

Thus

$$\alpha \left(\frac{K^*}{N}\right)^{\alpha-1} = \frac{1}{\beta^c} + \delta - 1$$

$$K^* = \left(\frac{\frac{1}{\beta^c} + \delta - 1}{\alpha}\right)^{\frac{1}{\alpha-1}} N$$

$$K^{**} = \left(\frac{\frac{1}{\beta^c} + \delta - 1}{\alpha^2}\right)^{\frac{1}{\alpha-1}} N = \left(\frac{1}{\alpha}\right)^{\frac{1}{\alpha-1}} K^* = \alpha^{\frac{1}{1-\alpha}} K^* < K^*$$

since $\alpha < 1$. For $\alpha = \frac{1}{3}$, $\alpha^{\frac{1}{1-\alpha}} = 0.19245$. Suppose $\beta^c = .96^{30} = 0.2938$ and $\delta = 1$. Then if $N = 1$,

$$K^* = (\alpha\beta^c)^{\frac{1}{1-\alpha}} = 0.0306$$

$$Y^* = (K^*)^\alpha = 0.3130$$

$$w^* = (1 - \alpha) \left(\frac{K^*}{N}\right)^\alpha = (1 - \alpha)Y^* = 0.2086$$

$$R^* = \alpha \left(\frac{K^*}{N} \right)^{\alpha-1} = 3.403$$

$$R^* K^* = 0.1043.$$

Meanwhile

$$K^{**} = (\alpha^2 \beta^c)^{\frac{1}{1-\alpha}} = 0.0059$$

$$Y^{**} = (K^{**})^\alpha = 0.1807$$

$$w^{**} = (1 - \alpha) \left(\frac{K^{**}}{N} \right)^\alpha = (1 - \alpha) Y^{**} = 0.1205$$

$$R^{**} = \alpha \left(\frac{K^{**}}{N} \right)^{\alpha-1} = 10.209$$

$$R^{**} K^{**} = 0.0602$$

In the steady state, the capitalist might be better off if he was a price-taker. However, he presumably cannot get there without losing out during the transition.

If $K^{**}/Y^{**} = 0.1$ for a 30-year period,

$$\frac{K^{**}}{Y^{**}} = \frac{(\alpha^2 \beta^c)^{\frac{1}{1-\alpha}}}{(\alpha^2 \beta^c)^{\frac{\alpha}{1-\alpha}}} = \alpha^2 \beta^c,$$

so if $\alpha = \frac{1}{3}$,

$$\beta^c = \frac{0.1}{(1/3)^2} = 9(0.1) = 0.9,$$

so $\beta_{ann}^c = (0.9)^{1/30} = 0.996$.

The reason efficiency does not arise is somewhat the reverse of Marx's surplus value argument. If capitalists were price-takers, they would be getting exploited by the laborers in the same way that Marx argues that laborers get exploited. When a capitalist increases his capital so the marginal product of capital equals his discount rate, he does not get the full benefit of all those incremental increases in capital because he only gets a fraction of total output.

A solution to this problem presumably cannot be obtained by incentivizing the capitalist to produce more capital. The solution is a more egalitarian distribution of wealth so no one has pricing power.

So, for example, suppose production is Cobb-Douglas and there is no depreciation. Optimal consumption will be such that the value of giving up the last unit of consumption equals the discounted value of additional consumption next period. A price-taker will behave as though the production function is linear when in fact it is concave.

Consider what happens in a two-period model when

$$v_T(K_T) = \ln(R(K_T)K_T)$$

and $\delta = 1$ so

$$R(K) = \alpha K^{\alpha-1}.$$

Thus

$$\begin{aligned}
v_T(K_T) &= \ln(\alpha K_T^\alpha). \\
v_{T-1}(K_{T-1}) &= \ln(C_{T-1}) + \beta^c \ln(\alpha K_T^\alpha) \\
C_{T-1} + K_T &= \alpha K_{T-1}^\alpha \\
L_{T-1}(C_{T-1}) &= \ln(C_{T-1}) + \beta^c \ln(\alpha(\alpha K_{T-1}^\alpha - C_{T-1})^\alpha) \\
&= \ln(C_{T-1}) + \beta^c \ln(\alpha) + \alpha\beta^c \ln(\alpha K_{T-1}^\alpha - C_{T-1}) \\
L'_{T-1}(C_{T-1}) &= \frac{1}{C_{T-1}} - \frac{\alpha\beta^c}{\alpha K_{T-1}^\alpha - C_{T-1}} = 0 \\
\alpha K_{T-1}^\alpha - C_{T-1} &= \alpha\beta^c C_{T-1} \\
C_{T-1} &= \frac{\alpha K_{T-1}^\alpha}{1 + \alpha\beta^c} \\
K_T = \alpha K_{T-1}^\alpha - \frac{\alpha K_{T-1}^\alpha}{1 + \alpha\beta^c} &= \frac{(\alpha + \alpha^2\beta^c)K_{T-1}^\alpha - \alpha K_{T-1}^\alpha}{1 + \alpha\beta^c} = \frac{\alpha^2\beta^c}{1 + \alpha\beta^c} K_{T-1}^\alpha.
\end{aligned}$$

If on the other hand,

$$\begin{aligned}
v_T(K_T) &= \ln(R_T K_T) \\
L_{T-1}(C_{T-1}) &= \ln(C_{T-1}) + \beta^c \ln(R_T(R_{T-1}K_{T-1} - C_{T-1})) \\
L'_{T-1}(C_{T-1}) &= \frac{1}{C_{T-1}} - \frac{\beta^c}{R_{T-1}K_{T-1} - C_{T-1}} = 0 \\
R_{T-1}K_{T-1} - C_{T-1} &= \beta^c C_{T-1} \\
C_{T-1} &= \frac{R_{T-1}K_{T-1}}{1 + \beta^c} = \frac{\alpha K_{T-1}^\alpha}{1 + \beta^c} \\
K_T = R_{T-1}K_{T-1} - \frac{R_{T-1}K_{T-1}}{1 + \beta^c} &= \frac{\beta^c}{1 + \beta^c} R_{T-1}K_{T-1} = \frac{\alpha\beta^c}{1 + \beta^c} K_{T-1}^\alpha
\end{aligned}$$

The price-taker ignores the curvature in the production function.

If there is no separation into labor and capital, then the problem becomes for a price taker,

$$\begin{aligned}
v_T(K_T) &= \ln(w_T + R_T K_T) \\
L_T(C_{T-1}) &= \ln(C_{T-1}) + \beta \ln(w_T + R_T(w_{T-1} + R_{T-1}K_{T-1} - C_{T-1})) \\
L'_T(C_{T-1}) &= \frac{1}{C_{T-1}} - \frac{\beta R_T}{w_T + R_T w_{T-1} + R_T R_{T-1} K_{T-1} - R_T C_{T-1}} = 0 \\
w_T + R_T w_{T-1} + R_T R_{T-1} K_{T-1} - R_T C_{T-1} &= \beta R_T C_{T-1} \\
w_{T-1} + \frac{w_T}{R_T} + R_{T-1} K_{T-1} &= (1 + \beta) C_{T-1} \\
C_{T-1} &= \frac{1}{1 + \beta} \left[w_{T-1} + \frac{w_T}{R_T} + R_{T-1} K_{T-1} \right],
\end{aligned}$$

which in equilibrium is

$$C_{T-1} = \frac{1}{1+\beta} \left[K_{T-1}^\alpha + \frac{(1-\alpha)K_T^\alpha}{\alpha K_T^{\alpha-1}} \right]$$

$$C_{T-1} = \frac{1}{1+\beta} \left[K_{T-1}^\alpha + \frac{1-\alpha}{\alpha} K_T \right]$$

Without price taking, the problem is

$$v_T(K_T) = \ln(K_T^\alpha) = \alpha \ln(K_T)$$

$$L_T(C_{T-1}) = \ln(C_{T-1}) + \alpha\beta \ln(K_{T-1}^\alpha - C_{T-1})$$

$$L'_T(C_{T-1}) = \frac{1}{C_{T-1}} - \frac{\alpha\beta}{K_{T-1}^\alpha - C_{T-1}} = 0$$

$$K_{T-1}^\alpha - C_{T-1} = \alpha\beta C_{T-1}$$

$$C_{T-1} = \frac{K_{T-1}^\alpha}{1+\alpha\beta}$$

$$K_T = K_{T-1}^\alpha - C_{T-1} = \frac{\alpha\beta}{1+\alpha\beta} K_{T-1}^\alpha$$

$$\begin{aligned} \frac{1}{1+\beta} \left[K_{T-1}^\alpha + \frac{1-\alpha}{\alpha} K_T \right] &= \frac{1}{1+\beta} \left[K_{T-1}^\alpha + \frac{1-\alpha}{\alpha} \frac{\alpha\beta}{1+\alpha\beta} K_{T-1}^\alpha \right] \\ &= \frac{1}{1+\beta} \left[1 + \frac{\beta - \alpha\beta}{1+\alpha\beta} \right] K_{T-1}^\alpha \\ &= \frac{1}{1+\beta} \frac{1+\alpha\beta + \beta - \alpha\beta}{1+\alpha\beta} K_{T-1}^\alpha \\ &= \frac{1}{1+\alpha\beta} K_{T-1}^\alpha = C_{T-1} \end{aligned}$$

So there is no difference in equilibrium.

When the capitalist and the laborer are one person, so he still gets the benefits to the wage that comes from saving more, the linear approximation works. In the segregated economy, it does not.

Suppose we have n types of agents with endowments e_t^i for $t = 0, 1$ that are endowed with k_0^i capital. Let ρ_i be the frequency of the i th type so

$$\sum_{i=1}^n \rho_i = 1.$$

$$N_t = \sum_{i=1}^n \rho_i e_t^i$$

$$K_t = \sum_{i=1}^n \rho_i k_t^i.$$

Agents of type i maximize

$$U_i = u_i(c_0^i) + \beta^i u_i(c_1^i).$$

subject to

$$c_t^i + k_{t+1}^i = w_t e_t^i + R_t k_t^i \quad t = 0, 1,$$

where $k_2^i = 0$ for $i = 1, \dots, n$. In equilibrium,

$$w_t = F_N(K_t, N_t)$$

$$R_t = F_K(K_t, N) + 1 - \delta,$$

where F has constant returns to scale.

$$C_t = \sum_{i=1}^n \rho c_t^i$$

$$\begin{aligned} C_t &= \sum_{i=1}^n \rho [w_t e_t^i + R_t k_t^i - k_{t+1}^i] = w_t \sum_{i=1}^n \rho e_t^i + R_t \sum_{i=1}^n \rho k_t^i - \sum_{i=1}^n \rho k_{t+1}^i \\ &= w_t N_t + R_t K_t - K_{t+1} \\ &= F_N(K_t, N_t) N_t + F_K(K_t, N) K_t + (1 - \delta) K_t - K_{t+1} \end{aligned}$$

$$C_t = F(K_t, N_t) + (1 - \delta) K_t - K_{t+1}.$$

Let

$$I_t = K_{t+1} - (1 - \delta) K_t.$$

Then

$$C_t + I_t = F(K_t, N_t).$$

$$L_i = u_i(c_0^i) + \beta^i u_i(c_1^i) + \lambda_0^i [w_0 e_0^i + R_0 k_0^i - c_0^i - k_1^i] + \lambda_1^i [w_1 e_1^i + R_1 k_1^i - c_1^i]$$

$$\frac{\partial L_i}{\partial c_0^i} = u_i'(c_0^i) - \lambda_0^i = 0$$

$$\frac{\partial L_i}{\partial c_1^i} = \beta^i u_i'(c_1^i) - \lambda_1^i = 0$$

$$\frac{\partial L}{\partial k_1^i} = \lambda_1^i R_1 - \lambda_0^i = 0$$

$$u_i'(c_0^i) = \beta^i R_1 u_i'(c_1^i).$$

Thus

$$u_i'(c_0^i) = \beta^i [F_K(K_1, N_1) + 1 - \delta] u_i'(c_1^i)$$

$$c_1^i = w_1 e_1^i + R_1 k_1^i$$

$$k_1^i = \frac{c_1^i - w_1 e_1^i}{R_1}$$

$$c_0^i + \frac{c_1^i - w_1 e_1^i}{R_1} = w_0 e_0^i + R_0 k_0^i$$

$$c_0^i + \frac{c_1^i}{R_1} = w_0 e_0^i + \frac{w_1 e_1^i}{R_1} + R_0 k_0^i$$

A social planner will maximize

$$U = \sum_{i=1}^n \xi_i [u_i(c_0^i) + \beta^i u_i(c_1^i)]$$

subject to

$$\sum_{i=1}^n \rho c_t^i + K_{t+1} - (1 - \delta)K_t = F(K_t, N_t)$$

where

$$K_0 = \sum_{i=1}^n \rho_i k_0^i$$

and $K_2 = 0$.

$$\begin{aligned} L_s = & \sum_{i=1}^n \xi_i [u_i(c_0^i) + \beta^i u_i(c_1^i)] + \mu_0 \left[F(K_0, N_0) - \sum_{i=1}^n \rho c_0^i - K_1 + (1 - \delta)K_0 \right] \\ & + \mu_1 \left[F(K_1, N_1) - \sum_{i=1}^n \rho c_1^i + (1 - \delta)K_1 \right] \end{aligned}$$

$$\frac{\partial L_s}{\partial c_0^i} = \xi_i u_i'(c_0^i) - \rho_i \mu_0 = 0$$

$$\frac{\partial L_s}{\partial c_1^i} = \xi_i \beta^i u_i'(c_1^i) - \rho_i \mu_1 = 0$$

$$\frac{\partial L_s}{\partial K_1} = -\mu_0 + \mu_1 [F_K(K_1, N_1) + 1 - \delta] = 0$$

$$\xi_i u_i'(c_0^i) = \rho_i \mu_0 = \rho_i \mu_1 [F_K(K_1, N_1) + 1 - \delta] = \xi_i u_i'(c_1^i) \beta^i [F_K(K_1, N_1) + 1 - \delta].$$

But in equilibrium

$$u_i'(c_0^i) = \beta^i [F_K(K_1, N_1) + 1 - \delta] u_i'(c_1^i).$$

We can write

$$c_1^i = \chi_i(c_0^i, \xi_i, K_1)$$

The key here is that the non-price taker's problem is not equivalent to a social planner's problem in the segregated economy model. The social planner has to be able to allocate all the output. The price-making capitalist only has access to the return from capital.

Suppose that we have a representative-agent with log utility and $\delta = 1$. For an agent who lives T periods, suppose

$$v_T(K) = A_T \ln(K) + D_T$$

where $A_1 = \alpha$ and $D_1 = 0$. The Bellman equation is

$$v_{T+1}(K) = \max_C \ln(C) + \beta A_T \ln(K^\alpha - C).$$

Thus

$$L_{T+1}(C) = \ln(C) + \beta A_T \ln(K^\alpha - C)$$

$$L'_{T+1}(C) = \frac{1}{C} - \frac{\beta A_T}{K^\alpha - C} = 0$$

$$\frac{1}{C} = \frac{\beta A_T}{K^\alpha - C}$$

$$K^\alpha - C = \beta A_T C$$

$$C = \frac{K^\alpha}{1 + \beta A_T}.$$

$$K' = K^\alpha - \frac{K^\alpha}{1 + \beta A_T} = \frac{\beta A_T}{1 + \beta A_T} K^\alpha$$

$$v_{T+1}(K) = \ln\left(\frac{K^\alpha}{1 + \beta A_T}\right) + \beta A_T \ln\left(\frac{\beta A_T}{1 + \beta A_T} K^\alpha\right) + \beta D_T$$

$$v_{T+1}(K) = \alpha(1 + \beta A_T) \ln(K) + \beta A_T \ln\left(\frac{\beta A_T}{1 + \beta A_T}\right) - \ln(1 + \beta A_T) + \beta D_T$$

$$A_{T+1} = \alpha(1 + \beta A_T)$$

$$D_{T+1} = \beta A_T \ln\left(\frac{\beta A_T}{1 + \beta A_T}\right) - \ln(1 + \beta A_T) + \beta D_T$$

The A_T equation has the solution

$$A_T = \alpha \frac{1 - (\alpha\beta)^T}{1 - \alpha\beta}$$

$$A_{T+1} = \alpha + \alpha^2 \beta \frac{1 - (\alpha\beta)^T}{1 - \alpha\beta} = \frac{\alpha - \alpha^2 \beta + \alpha^2 \beta - \alpha(\alpha\beta)^{T+1}}{1 - \alpha\beta} = \alpha \frac{1 - (\alpha\beta)^{T+1}}{1 - \alpha\beta}.$$

Thus the saving rate in this model is

$$s_{T+1} = \frac{\beta A_T}{1 + \beta A_T} = \frac{\alpha \beta \frac{1 - (\alpha\beta)^T}{1 - \alpha\beta}}{1 + \alpha \beta \frac{1 - (\alpha\beta)^T}{1 - \alpha\beta}} = \frac{\alpha \beta (1 - (\alpha\beta)^T)}{1 - \alpha\beta + \alpha\beta - (\alpha\beta)^{T+1}} = \alpha \beta \frac{1 - (\alpha\beta)^T}{1 - (\alpha\beta)^{T+1}} < \alpha \beta,$$

which converges to $\alpha\beta$ in the limit as $T \rightarrow \infty$. We need $\alpha\beta < 1$ in order for the model to converge to K^* as $T \rightarrow \infty$. Thus we need both $\alpha < 1$ and $\beta < 1$.

Under the Golden Rule, the saving rate will be α , so the saving rate is always less than α , and the Golden Rule cannot be achieved in this model.

With more general depreciation rates,

$$R = \alpha K^{\alpha-1} + 1 - \delta = 1,$$

so

$$\alpha K^{\alpha-1} = \delta$$

$$\alpha \frac{Y}{K} = \delta,$$

so

$$\frac{K}{Y} = \frac{\alpha}{\delta}.$$

Sadly, the analytic solution also goes away if we introduce the capital tax so we cannot consider the political economy question. This is because we would need $(1 - \tau_k)\delta = 1$ to preserve the analytic solution, which would only be possible with more than 100% depreciation.

However, perturbation theory might work. Let $\varepsilon = 1 - \delta$. Then

$$v_T(K) = \ln \left(\left(\alpha \left(\frac{K}{N} \right)^{\alpha-1} + \varepsilon \right) K \right).$$

$$L_{T-1}(C) = \ln(C) + \beta \ln \left(\left(\alpha \left(\frac{K_T(C)}{N} \right)^{\alpha-1} + \varepsilon \right) K_T(C) \right)$$

where

$$K_{t+1}(C) = \left(\alpha \left(\frac{K_t}{N} \right)^{\alpha-1} + \varepsilon \right) K_t - C.$$

$$K'_{t+1}(C) = -1.$$

$$L'_{T-1}(C) = \frac{1}{C} - \frac{\beta}{\alpha \left(\frac{K_T(C)}{N} \right)^{\alpha-1} + \varepsilon} \frac{\alpha(\alpha-1)}{N} \left(\frac{K_T(C)}{N} \right)^{\alpha-2} - \frac{\beta}{K_T(C)} = 0$$

$$\frac{1}{C} - \frac{\beta}{\alpha \left(\frac{K_T(C)}{N} \right)^{\alpha-1}} \frac{\alpha(\alpha-1)}{N} \left(\frac{K_T(C)}{N} \right)^{\alpha-2} \left[1 - \frac{\varepsilon}{\alpha} \left(\frac{K_T(C)}{N} \right)^{1-\alpha} \right] - \frac{\beta}{K_T(C)} = 0$$

$$\frac{1}{C} - \frac{(\alpha-1)\beta}{N} \frac{N}{K_T(C)} \left[1 - \frac{\varepsilon}{\alpha} \left(\frac{K_T(C)}{N} \right)^{1-\alpha} \right] - \frac{\beta}{K_T(C)} = 0$$

$$\frac{1}{C} - \frac{\alpha\beta}{K_T(C)} - \frac{1-\alpha}{\alpha} \frac{\beta}{N} \left(\frac{K_T(C)}{N} \right)^{-\alpha} \varepsilon = 0$$

Let

$$G(K) = \left(\alpha \left(\frac{K}{N} \right)^{\alpha-1} + \varepsilon \right) K$$

$$\frac{1}{C_{T-1}} = \frac{\alpha\beta}{G(K_{T-1}) - C_{T-1}} + \frac{1-\alpha}{\alpha} \frac{\beta}{N} \left(\frac{G(K_{T-1}) - C_{T-1}}{N} \right)^{-\alpha} \varepsilon$$

Let

$$C_{T-1} = c_{T-1}^0 + c_{T-1}^1 \varepsilon$$

$$G(K_{T-1}) - c_{T-1}^0 - c_{T-1}^1 \varepsilon = \alpha\beta (c_{T-1}^0 + c_{T-1}^1 \varepsilon) + c_{T-1}^0 \beta \frac{1-\alpha}{\alpha} \left(\frac{G(K_{T-1}) - c_{T-1}^0}{N} \right)^{1-\alpha} \varepsilon + O(\varepsilon^2)$$

$$G(K_{T-1}) = (1 + \alpha\beta)c_{T-1}^0$$

$$c_{T-1}^0 = \frac{G(K_{T-1})}{1 + \alpha\beta}$$

$$(1 + \alpha\beta)c_{T-1}^1 = -c_{T-1}^0 \beta \frac{1-\alpha}{\alpha} \left(\frac{G(K_{T-1}) - c_{T-1}^0}{N} \right)^{1-\alpha}$$

$$c_{T-1}^1 = -\frac{1}{1 + \alpha\beta} \frac{G(K_{T-1})}{1 + \alpha\beta} \beta \frac{1-\alpha}{\alpha} \left(\frac{\alpha\beta G(K_{T-1})}{1 + \alpha\beta} \right)^{1-\alpha}$$

$$c_{T-1}^1 = -\frac{1-\alpha}{\alpha + \alpha^2\beta} \left(\frac{\alpha\beta G(K_{T-1})}{1 + \alpha\beta} \right)^{2-\alpha}$$

Note that if equilibria are not efficient, the business cycle likely exacerbates that efficiency, creating a role for policy intervention.

In the context of the full model, the analytic problem has $\delta = 1$, $\gamma_c = 1$, $\tau^k = 0$, and

$$N = \frac{\mu\eta}{1 + \eta}.$$

The Bellman equation is

$$v(k_t^c, K_t) = \max \ln(c_t^c) + \beta^c v(k_{t+1}^c, K_{t+1})$$

subject to

$$c_t^c + k_{t+1}^c = R(K_t)k_t^c$$

$$K_{t+1} = (1 - \mu)k_{t+1}^c \tag{79}$$

$$R(K) = \alpha \left(\frac{K}{N} \right)^{\alpha-1}. \tag{80}$$

Let us guess that

$$v(k^c, K) = A \ln \left(\alpha \left(\frac{K}{N} \right)^{\alpha-1} k^c \right) + D. \tag{81}$$

$$\begin{aligned} L(c) &= \ln(c) + \beta^c A \ln \left(\alpha \left(\frac{(1-\mu)(R(K)k^c - c)}{N} \right)^{\alpha-1} (R(K)k^c - c) \right) + \beta^c D \\ &= \ln(c) + \alpha\beta^c A \ln(R(K)k^c - c) + \beta^c A \ln \left(\alpha \left(\frac{1-\mu}{N} \right)^{\alpha-1} \right) + \beta^c D \end{aligned}$$

$$L'(c) = \frac{1}{c} - \frac{\alpha\beta^c A}{R(K)k^c - c} = 0$$

$$R(K)k^c - c - \alpha\beta^c A c = 0$$

$$c(k^c, K) = \frac{R(K)k^c}{1 + \alpha\beta^c A}$$

$$R(K)k^c - c = R(K)k^c - \frac{R(K)k^c}{1 + \alpha\beta^c A} = \frac{\alpha\beta^c A}{1 + \alpha\beta^c A} R(K)k^c$$

$$v(k^c, K) = \ln\left(\frac{R(K)k^c}{1 + \alpha\beta^c A}\right) + \alpha\beta^c A \ln\left(\frac{\alpha\beta^c A}{1 + \alpha\beta^c A} R(K)k^c\right) + \beta^c A \ln\left(\alpha\left(\frac{1-\mu}{N}\right)^{\alpha-1}\right) + \beta^c D$$

$$\begin{aligned} A \ln(R(K)k^c) + D &= (1 + \alpha\beta^c A) \ln(R(K)k^c) - \ln(1 + \alpha\beta^c A) + \alpha\beta^c A \ln\left(\frac{\alpha\beta^c A}{1 + \alpha\beta^c A}\right) \\ &\quad + \beta^c A \ln\left(\alpha\left(\frac{1-\mu}{N}\right)^{\alpha-1}\right) + \beta^c D \end{aligned}$$

$$A = 1 + \alpha\beta^c A$$

$$A = \frac{1}{1 - \alpha\beta^c}$$

$$\frac{\alpha\beta^c A}{1 + \alpha\beta^c A} = \frac{\frac{\alpha\beta^c}{1 - \alpha\beta^c}}{1 + \frac{\alpha\beta^c}{1 - \alpha\beta^c}} = \alpha\beta^c$$

$$1 + \alpha\beta^c A = 1 + \frac{\alpha\beta^c}{1 - \alpha\beta^c} = \frac{1}{1 - \alpha\beta^c}$$

$$D = \frac{1}{1 - \beta^c} \left[\ln(1 - \alpha\beta^c) + \frac{\alpha\beta^c}{1 - \alpha\beta^c} \ln(\alpha\beta^c) + \frac{\beta^c}{1 - \alpha\beta^c} \left(\alpha\left(\frac{1-\mu}{N}\right)^{\alpha-1} \right) \right]$$

Thus the policy rule is

$$c(k^c, K) = (1 - \alpha\beta^c) \alpha \left(\frac{K}{N}\right)^{\alpha-1} k^c \quad (82)$$

$$k_{t+1}^c(k_t^c, K_t) = \alpha^2 \beta^c \left(\frac{K_t}{N}\right)^{\alpha-1} k_t^c \quad (83)$$

In equilibrium,

$$k_{t+1}^c(k_t^c) = \alpha^2 \beta^c \left(\frac{1-\mu}{N}\right)^{\alpha-1} (k_t^c)^\alpha$$

In the steady state,

$$\frac{K}{Y} = \frac{(1-\mu)k^c}{Y} = \frac{\alpha^2 \beta^c (1-\mu)^\alpha N^{1-\alpha} (k^c)^\alpha}{Y} = \frac{\alpha^2 \beta^c K^\alpha N^{1-\alpha}}{Y} = \alpha^2 \beta.$$

Note that if $\alpha = 1/3$,

$$\frac{\frac{K_{ps}}{Y_{ps}}}{\frac{K_{pt}}{Y_{pt}}} = \alpha$$

$$\frac{K_{ps}^{1-\alpha}}{K_{pt}^{1-\alpha}} = \alpha$$

$$\frac{K_{ps}}{K_{pt}} = \alpha^{\frac{1}{1-\alpha}}$$

$$\frac{w_{ps}}{w_{pt}} = \left(\frac{K_{ps}}{K_{pt}}\right)^\alpha = \alpha^{\frac{\alpha}{1-\alpha}} = \left(\frac{1}{3}\right)^{\frac{1}{2}} = 0.57735,$$

so in the analytic case we should get a 43% reduction in wages in the steady state.

There are two mechanisms by which price setting works. The first mechanism is the direct mechanism that if the capitalist saves more then he increases the aggregate capital stock too, resulting in a smaller rate of return. The indirect mechanism is that if the capitalist saves more, wages will be higher, so the laborer will save more.

In a two-period version of the model, only the direct mechanism appears. In that case,

$$v_T(k, K) = u_c(R(K)k).$$

$$\begin{aligned} k_T &= R_{T-1}(K_{T-1})k_{T-1} - c_{T-1}^c \\ K_T &= (1 - \tau_l)s_l\mu w(K_{T-1}) + (1 - \mu)k_T \end{aligned}$$

$$L(c_{T-1}^c) = u(c_{T-1}^c) + \beta^c u(R_T(K_T)k_T)$$

$$\frac{\partial}{\partial k_T}(R_T(K_T)k_T) = R_T(K_T) + (1 - \mu)R_T'(K_T)k_T = R_T(K_T) \left[1 + \frac{K_T R_T'(K_T)}{R_T(K_T)} \frac{(1 - \mu)k_T}{K_T} \right]$$

Let

$$\varepsilon_t = \frac{K_t R_t'(K_t)}{R_t(K_t)} \tag{84}$$

be the elasticity of the rate of return on capital with respect to capital and let

$$\kappa_t = \frac{(1 - \mu)k_t}{K_t} \tag{85}$$

be the fraction of the capital stock owned by the capitalist. Then the Euler equation becomes

$$u_c'(c_{T-1}^c) - \beta^c R_T(K_T) [1 + \varepsilon_T \kappa_T] u_c'(c_T) = 0$$

$$\frac{u_c'(c_{T-1}^c)}{u_c'(c_T^c)} = \beta^c R_T(K_T) [1 + \varepsilon_T \kappa_T].$$

If the capitalist has CRRA utility, this simplifies to a result for the growth rate of consumption:

$$\frac{c_T^c}{c_{T-1}^c} = (\beta^c R_T(K_T)[1 + \varepsilon_T \kappa_T])^{1/\gamma_c}. \quad (86)$$

The elasticity ε_t is a function of the curvature of the production function:

$$\begin{aligned} \varepsilon_t &= \frac{(1 - \tau_t^k) K_t F_{KK}(K_t, N_t)}{1 + (1 - \tau_t^k)(F_K(K_t, N_t) - \delta)} \\ \varepsilon_t &= \frac{1}{1 + \frac{1 - (1 - \tau_t^k)\delta}{(1 - \tau_t^k)F_K(K_t, N_t)}} \frac{K_t F_{KK}(K_t, N_t)}{F_K(K_t, N_t)} \\ \varepsilon_t &= \frac{(1 - \tau_t^k)F_K(K_t, N_t)}{R(K_t)} \frac{K_t F_{KK}(K_t, N_t)}{F_K(K_t, N_t)} \end{aligned}$$

For the case of Cobb-Douglas production,

$$\begin{aligned} F_K(K_t, N_t) &= \alpha \left(\frac{K_t}{N_t} \right)^{\alpha-1} \\ F_{KK}(K_t, N_t) &= \frac{(\alpha - 1)\alpha}{K_t} \left(\frac{K_t}{N_t} \right)^{\alpha-1} \\ \varepsilon_t &= -(1 - \alpha) \frac{R(K_t) - (1 - (1 - \tau_t^k)\delta)}{R(K_t)} \end{aligned} \quad (87)$$

The more the capitalist owns of the capital stock, the bigger the deviation there will be between price-taking and price-setting behavior. Likewise, if $\tau_t^k = 0$ and $\delta = 1$, then $\varepsilon_t = -(1 - \alpha)$. If $\beta_l = 0$, then we get

$$\frac{c_T^c}{c_{T-1}^c} = (\alpha \beta^c R_T(K_T))^{1/\gamma_c}. \quad (88)$$

That is presumably when the effect is largest. When $\gamma_c = 1$, this also corresponds to the analytic case.

The deviation is larger when the capitalist owns more of the capital stock, but also when the share of capital is low. The production function is more curved when the share of capital is low.

Note, however, that if we define

$$\frac{c_T^c}{c_{T-1}^c} = (\beta^c R_T(K_T))^{1/\tilde{\gamma}_c}$$

so

$$\begin{aligned} (\beta^c R_T(K_T))^{1/\tilde{\gamma}_c} &= (\beta^c R_T(K_T)[1 + \varepsilon_T \kappa_T])^{1/\gamma_c} \\ \frac{1}{\tilde{\gamma}_c} \ln [\beta^c R_T(K_T)] &= \ln [\beta^c R_T(K_T)] + \frac{1}{\gamma_c} \ln(1 + \varepsilon_T \kappa_T) \end{aligned}$$

$$\begin{aligned}\tilde{\gamma}_c &= \frac{\ln[\beta^c R_T(K_T)]}{\ln[\beta^c R_T(K_T)] + \frac{1}{\gamma_c} \ln(1 + \varepsilon_T \kappa_T)} = \frac{\gamma_c}{\gamma_c + \frac{\ln(1 + \varepsilon_T \kappa_T)}{\ln[\beta^c R_T(K_T)]}} \\ \frac{1}{\tilde{\gamma}_c} &= 1 + \frac{1}{\gamma_c} \frac{\ln(1 + \varepsilon_T \kappa_T)}{\ln[\beta^c R_T(K_T)]}\end{aligned}$$

Note that $\varepsilon_T < 0$, so the denominator is smaller than the numerator. Thus the apparent elasticity of intertemporal substitution will be less than $\frac{1}{\gamma_c}$.

6 Price-Taking Behavior

For comparison, if the capitalist was a price taker, then we would have

$$\begin{aligned}L(c_{T-1}^c) &= u(c_{T-1}^c) + \beta^c u(R_T(R_{T-1}k_{T-1}^c - c_{T-1}^c)) \\ L'(c_{T-1}^c) &= (c_{T-1}^c)^{-\gamma_c} - \beta^c R_T(R_T(R_{T-1}k_{T-1}^c - c_{T-1}^c))^{-\gamma_c} = 0 \\ c_{T-1}^c &= (\beta^c R_T)^{-1/\gamma_c} (R_T R_{T-1} k_{T-1}^c - R_T c_{T-1}^c)\end{aligned}$$

Let

$$\begin{aligned}\phi_t^c &= (\beta^c R_t^{1-\gamma_c})^{-1/\gamma_c} \\ (1 + \phi_T^c) c_{T-1}^c &= \phi_T^c R_{T-1} k_{T-1}^c \\ c_{T-1}^c &= \frac{\phi_T^c}{1 + \phi_T^c} R_{T-1} k_{T-1}^c \\ k_T^c &= \frac{1}{1 + \phi_T^c} R_{T-1} k_{T-1}^c \\ K_T &= \mu s_l (1 - \tau_l) w(K_{T-1}) + (1 - \mu) \frac{1}{1 + \phi_T^c(K_T)} R_{T-1}(K_{T-1}) k_{T-1}^c\end{aligned}$$

For the case where $\gamma_c = 1$, we are done. More generally, we would have to solve this for K_T .

More generally, let

$$v_t(k_t^c) = M_t \ln(k_t^c) + D_t,$$

where $M_T = 1$.

$$\begin{aligned}L_t(c_t^c) &= \ln(c_t^c) + \beta^c [M_{t+1} \ln(R_t k_t^c - c_t^c) + D_{t+1}] \\ \frac{1}{c_t^c} - \frac{\beta^c M_{t+1}}{R_t k_t^c - c_t^c} &= 0 \\ R_t k_t^c - c_t^c &= \beta^c M_{t+1} c_t^c \\ c_t^c &= \frac{R_t k_t^c}{1 + \beta^c M_{t+1}} \\ k_{t+1}^c &= \frac{\beta^c M_{t+1}}{1 + \beta^c M_{t+1}} R_t k_t^c\end{aligned}$$

$$\begin{aligned}
v_t(k_t^c) &= \ln\left(\frac{R_t k_t^c}{1 + \beta^c M_{t+1}}\right) + \beta^c M_{t+1} \ln\left(\frac{\beta^c M_{t+1}}{1 + \beta^c M_{t+1}} R_t k_t^c\right) + \beta^c D_{t+1} \\
M_t \ln(k_t^c) &= (1 + \beta^c M_{t+1}) \ln(k_{t+1}^c) \\
M_t &= 1 + \beta^c M_{t+1} = \frac{1 - (\beta^c)^{T-t+1}}{1 - \beta^c} \\
M_{t+1} &= \frac{1 - (\beta^c)^{T-t}}{1 - \beta^c} \\
M_t &= 1 + \beta^c \frac{1 - (\beta^c)^{T-t}}{1 - \beta^c} = \frac{1 - \beta^c + \beta^c - (\beta^c)^{T-t+1}}{1 - \beta^c} = \frac{1 - (\beta^c)^{T-t+1}}{1 - \beta^c} \\
c_t^c &= \frac{1}{M_t} R_t k_t^c
\end{aligned}$$

So for example,

$$c_{T-2}^c = \frac{R_t k_t^c}{1 + \beta^c + (\beta^c)^2}$$

If $k_t^c = k_{t+1}^c$,

$$\frac{c_{T-1}^c}{c_{T-2}^c} = \frac{1 + \beta^c + (\beta^c)^2}{1 + \beta^c} = G$$

$$1 - G + (1 - G)\beta^c + (\beta^c)^2 = 0$$

$$\begin{aligned}
\beta^c &= \frac{G - 1 \pm \sqrt{(1 - G)^2 - 4(1 - G)}}{2} \\
&= \frac{G - 1 \pm \sqrt{G^2 - 2G + 1 - 4 + 4G}}{2} \\
&= \frac{G - 1 \pm \sqrt{(1 - G)(1 - G - 4)}}{2} \\
&= \frac{G - 1 \pm \sqrt{(G - 1)(G + 3)}}{2}
\end{aligned}$$

Since $G > 1$, $(G - 1)(G + 3) > (G - 1)^2$, so only the positive root is positive.

$$\beta^c = \frac{G - 1 + \sqrt{(G - 1)(G + 3)}}{2}$$

Interestingly, the price-setting capitalists end up with a smaller share of wealth than the price-taking capitalists.

When we converge to the infinite horizon, the price-taking capitalist's problem will be

$$v(k_t^c, K_t) = \max \ln(c_t^c) + \beta^c v(k_{t+1}^c, K_{t+1}) \quad (89)$$

subject to

$$k_{t+1}^c = R(K_t)k_t^c - c_t^c$$

where K_{t+1} is known. Let

$$\begin{aligned}
v(k_t^c, K_t) &= M \ln(k_t^c) + G(K_t) \\
\mathcal{L}(c_t^c) &= \ln(c_t^c) + \beta^c M \ln(R(K_t)k_t^c - c_t^c) + \beta^c G(K_{t+1}) \\
\frac{1}{c_t^c} &= \frac{\beta^c M}{R(K_t)k_t^c - c_t^c} = 0 \\
c_t^c &= \frac{R(K_t)k_t^c}{1 + \beta^c M} \\
k_{t+1}^c &= \frac{\beta^c M}{1 + \beta^c M} R(K_t)k_t^c \\
v(k_t^c, K_t) &= \ln\left(\frac{R(K_t)k_t^c}{1 + \beta^c M}\right) + \beta^c M \ln\left(\frac{\beta^c M}{1 + \beta^c M} R(K_t)k_t^c\right) + \beta^c G(K_{t+1}) \\
M \ln(k_t^c) &= (1 + \beta^c M) \ln(k_t^c) \\
M &= 1 + \beta^c M \\
M &= \frac{1}{1 - \beta^c} \\
1 + \beta^c M &= 1 + \frac{\beta^c}{1 - \beta^c} = \frac{1}{1 - \beta^c} \\
c_t^c &= (1 - \beta^c) R(K_t)k_t^c \tag{90}
\end{aligned}$$

In the steady state,

$$k^c = R(K)k^c - c^c$$

so

$$c^c = (R(K) - 1)k^c,$$

meaning the capitalist just consumes the interest on his capital.

7 Calibration

If we assume $e_0 = 1$, $e_1 = 0$, and $\gamma^l = \gamma^c = 1$, then our remaining parameters are μ , β^c , β^l , η , α , and δ . We can calibrate $\alpha = 1/3$. The capitalists should be 1% of the population. The total population is $2\mu + 1 - \mu = 1 + \mu$, so we want

$$\frac{1 - \mu}{1 + \mu} = 0.01$$

$$\begin{aligned}
1 - \mu &= 0.01(1 + \mu) \\
0.99 &= 1.01\mu \\
\mu &= \frac{0.99}{1.01} = 0.9802.
\end{aligned}$$

Then we can calibrate η so that $1 - l_0 = \frac{128}{168}$, where we assume laborers work a 40-hour week. To start with a no-government model, we can calibrate δ to match $\frac{C}{Y} = 0.75$ and β^c and β^l to match $\frac{K}{Y} = 3$ and the fraction of wealth owned by capitalists, $\kappa = \frac{(1-\mu)k}{K} = 0.3$, as reported by Piketty (2014).

$$n = 1 - l_0 = \frac{1 + \beta^l}{1 + \beta^l \eta} \eta$$

$$(1 + \beta^l \eta)n = (1 + \beta^l)\eta$$

$$n = (1 + \beta^l(1 - n))\eta$$

$$\eta = \frac{n}{1 + \beta^l(1 - n)}$$

This may be an artifact of the special case here, but the steady state R seems to be largely independent of μ , so the laborer's welfare is also largely independent of μ . The capitalist's welfare, on the other hand, is very sensitive to μ . Note that in the price-taking case, $R^* = 1/\beta^c$, so the laborer's welfare is completely independent of μ .

In the steady state,

$$k^c = \frac{K - \mu(1 - \tau_l)sw(K)}{1 - \mu}$$

$$s = \frac{\beta^l \eta}{1 + \beta^l \eta}$$

Thus an equilibrium is only possible if

$$K > \mu(1 - \tau_l)sw(K).$$

If we fix $n = 1 - l$,

$$n = \frac{1 + \beta^l}{1 + \beta^l \eta} \eta$$

Thus we need

$$K > \mu(1 - \tau_l) \frac{\beta^l}{1 + \beta^l} nw(K)$$

$$\frac{K}{N} > (1 - \tau_l) \frac{\beta^l}{1 + \beta^l} (1 - \alpha) \left(\frac{K}{N} \right)^\alpha$$

$$\left(\frac{K}{N} \right)^{1-\alpha} > (1 - \tau_l) \frac{\beta^l}{1 + \beta^l} (1 - \alpha)$$

$$\frac{K}{Y} > (1 - \tau_l) \frac{\beta^l}{1 + \beta^l} (1 - \alpha)$$

$$1 + (\beta^l)^{-1} > (1 - \tau_l)(1 - \alpha) \frac{Y}{K}$$

$$\beta^l < \frac{1}{(1 - \tau_l)(1 - \alpha)\frac{Y}{K} - 1}. \quad (91)$$

In the no-tax equilibrium with $\frac{K}{Y} = 0.1$, we need

$$\beta^l < \frac{1}{\frac{2}{3}10 - 1} = 0.176.$$

On an annual basis, this upper bound is 0.9438.

If β^l is sufficiently high relative to β^c , $k_t^c \rightarrow 0$. If $k_t^c \rightarrow 0$, in the steady state

$$\begin{aligned} K^* &= \mu(1 - \tau_l)sw(K^*) \\ K^* &= \mu(1 - \tau_l)s(1 - \alpha) \left(\frac{K^*}{N} \right)^\alpha \\ (K^*)^{1-\alpha} &= \frac{\mu(1 - \tau_l)s(1 - \alpha)}{N^\alpha} \\ (K^*)^{\alpha-1} &= \frac{N^\alpha}{\mu(1 - \tau_l)s(1 - \alpha)} \\ \left(\frac{K^*}{N} \right)^{\alpha-1} &= \frac{N}{\mu(1 - \tau_l)s(1 - \alpha)} = \frac{\frac{1+\beta^l}{1+\beta^l\eta}\eta}{(1 - \tau_l)(1 - \alpha)\frac{\beta^l\eta}{1+\beta^l\eta}} = \frac{1}{(1 - \tau_l)(1 - \alpha)} \frac{1 + \beta^l}{\beta^l} \end{aligned}$$

$$\begin{aligned} R^* &= \alpha \left(\frac{K^*}{N} \right)^{\alpha-1} + 1 - \delta \\ &= \frac{1}{1 - \tau_l} \frac{\alpha}{1 - \alpha} \frac{1 + \beta^l}{\beta^l} + 1 - \delta. \end{aligned}$$

In the limit where k^c is very small,

$$k_{t+1}^c \approx \beta^c R^* k_t^c.$$

Thus if $\beta^c R^* < 1$, we can get $k_t^c \rightarrow 0$. This will happen when

$$\begin{aligned} \beta^c \left[\frac{1}{1 - \tau_l} \frac{\alpha}{1 - \alpha} \frac{1 + \beta^l}{\beta^l} + 1 - \delta \right] &< 1. \\ \frac{1}{1 - \tau_l} \frac{\alpha}{1 - \alpha} \frac{1 + \beta^l}{\beta^l} + 1 - \delta &< (\beta^c)^{-1} \\ \frac{1}{1 - \tau_l} \frac{\alpha}{1 - \alpha} \left(1 + (\beta^l)^{-1} \right) &< (\beta^c)^{-1} - 1 + \delta \\ 1 + (\beta^l)^{-1} &< (1 - \tau_l) \frac{1 - \alpha}{\alpha} [(\beta^c)^{-1} - 1 + \delta] \\ \beta^l > \bar{\beta}^l &= \frac{1}{(1 - \tau_l) \frac{1 - \alpha}{\alpha} [(\beta^c)^{-1} - 1 + \delta] - 1} \end{aligned} \quad (92)$$

For $\beta^c = 1$, this upper bound is

$$\frac{1}{(1 - \tau_l) \frac{1 - \alpha}{\alpha} \delta - 1}.$$

If $\alpha = 1/3$,

$$(1 - \tau_l) \frac{1 - \alpha}{\alpha} \delta < 2$$

so the upper bound will be bigger than 1, keeping in mind that $\beta^l > 1$ is possible, but we would need $\beta^l > \beta^c$.

Note that this is not a proof that if $\beta^l > \bar{\beta}^l$ then $k_t^c \rightarrow 0$. But if $\beta^l < \bar{\beta}^l$, $k_t^c \rightarrow 0$ is not possible.

The Bellman equation in the infinite horizon for this special case simplifies to

$$\frac{[1 - \beta^c R(K^*)]}{c_c^*} = (1 - \mu) \beta^c \frac{\partial v}{\partial K}(k_c^*, K^*). \quad (93)$$

I suspect that c_c^* is proportional to $(1 - \mu)^{-1}$ and $\frac{\partial v}{\partial K}(k_c^*, K^*)$ reduces to a function of K^* , so that (93) determines K^* independently of μ .

In a steady state, we must have

$$K^* = \mu(1 - \tau_l)sw(K^*) + (1 - \mu)k_c^*.$$

We must also have

$$c_c^* = (R(K^*) - 1)k_c^*.$$

$$s = \frac{\beta^l \eta}{1 + \beta^l \eta}$$

$$N = \mu \frac{1 + \beta^l}{1 + \beta^l \eta} \eta$$

$$\frac{K^*}{N} = \frac{1 - \tau_l}{\frac{1 + \beta^l}{1 + \beta^l \eta} \eta} \frac{\beta^l \eta}{1 + \beta^l \eta} (1 - \alpha) \left(\frac{K^*}{N} \right)^\alpha + (1 - \mu) \frac{k_c^*}{N}$$

$$\frac{K^*}{N} = (1 - \tau_l)(1 - \alpha) \frac{\beta^l}{1 + \beta^l} \left(\frac{K^*}{N} \right)^\alpha + (1 - \mu) \frac{k_c^*}{N}$$

$$c_c^* = (R(K^*) - 1) \frac{N}{1 - \mu} \left[\frac{K^*}{N} - (1 - \tau_l)(1 - \alpha) \frac{\beta^l}{1 + \beta^l} \left(\frac{K^*}{N} \right)^\alpha \right]$$

$$v_T(k_T, K_T) = \ln(R(K_T)k_T)$$

If

$$v(k, K) = \tilde{V}(k, R(K), w(K)) = V\left(k, \frac{K}{N}\right),$$

then

$$\frac{\partial v}{\partial K} = \frac{\partial V}{\partial \left(\frac{K}{N}\right)} \left(k, \frac{K}{N}\right) \frac{1}{N}$$

Variable	PT	PS	% Change
Y	0.0823	0.0761	-7.547%
$(K/Y)_{ann}$	3.488	2.974	14.728%
r_{ann}	3.627%	4.166%	14.857%
κ	0.3228	0.2059	-36.229%
c^c	2.9571	1.8679	-36.834%
U^c	1.6513	0.9516	-36.834%
c_0^l	0.0488	0.0451	-7.547%
c_1^l	0.0189	0.0204	8.018%
U^l	-0.9803	-0.9950	-5.841%

Table 1: Steady-State Observables with both price-taking and price-setting capitalists for the baseline calibration.

Within the accuracy of my current approximations, $\frac{\partial v}{\partial K}(k^*, K^*)$ does appear to be proportional to N^{-1} . If that is the case, then this conjecture could be correct. Likewise, c_c^* appears to be proportional to $\frac{\mu}{1-\mu}$.

For the calibration described above, steady state observables for both the price-taking and price-setting equilibria are given in Table 1. The percent change from the price-taking to the price-setting equilibrium is also reported.¹³

Note that both the laborer and the capitalist suffer utility losses in the price-setting steady state. Indeed, the utility loss of the capitalist is much larger than for the worker. Pareto-improving transitions do exist that can take both capitalists and laborers from the price-setting steady state to the price-taking steady without anyone losing. However, these require interventions that go outside of what is possible in free markets. Once they are in the price-taking equilibrium again, the capitalists would have the same incentive to take advantage of their market power to get a short-term utility boost. This is shown in Fig.1, where the consumption of the capitalist both in the price-taking steady state and during the price-setting equilibrium transition from the price-taking steady state to the price-setting steady state are both displayed. The capitalist's consumption is only higher in the period when he first deviates from the price-taking equilibrium, but the utility loss from lower consumption in later periods is discounted. The magnitude of the difference in utility between the two equilibria, both per period and cumulative, is also plotted in Fig. 1. How the laborer's consumption varies during the transition is shown in Fig. 2. Notice that the initial cohort of old workers benefits from the capitalists' price setting just as the capitalists do since the rate of return on their capital increases. They are already retired, so the effect on wages does not impact them. For everyone else, the resulting decrease in wages causes a loss of utility that eventually reaches the equivalent of 6% of consumption in the price-taking steady state.

In Fig. 3 we show for price-taking equilibria calibrated so $K/Y = 3$ (in an-

¹³For U^c and U^l , the reported percentage change is actually in the percentage change in equivalent consumption from the price-taking equilibrium.

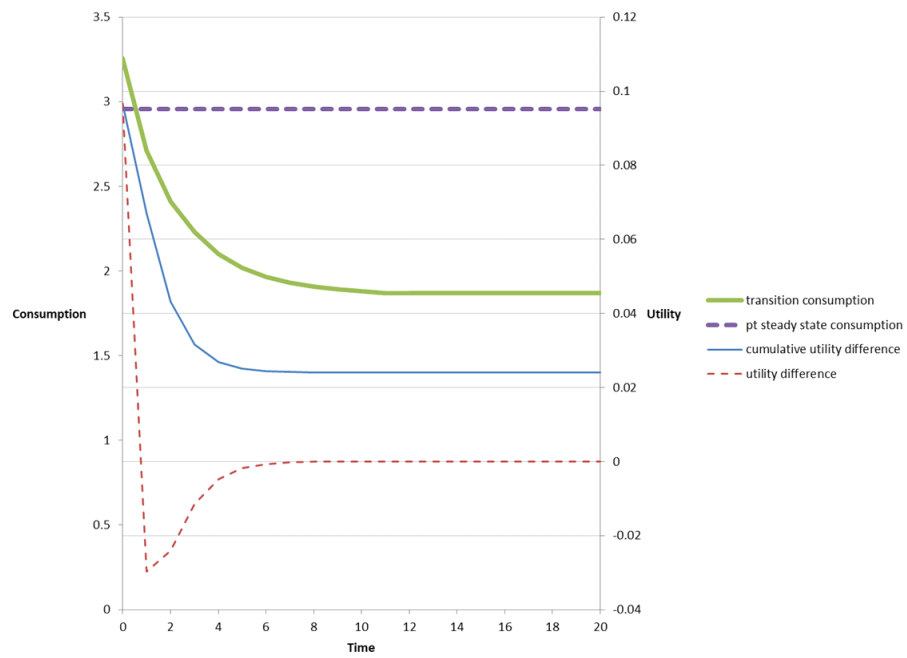


Figure 1: Capitalist consumption and utility during transition from price-taking to price-setting steady state for the no-tax baseline calibration

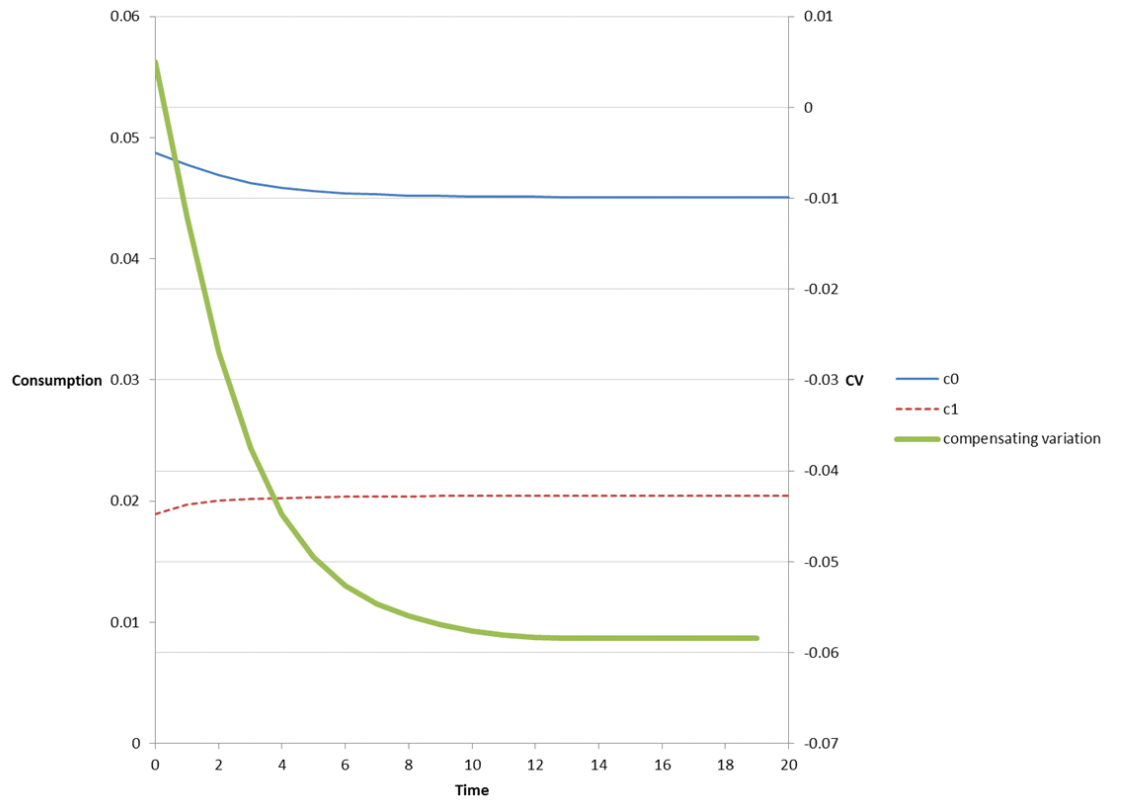


Figure 2: Consumptions and compensating variation for laborers during the equilibrium transition from the price-taking steady state to the price-setting steady state.

nual terms) how output, capital, and the laborer's utility (in terms of equivalent consumption) would decrease if capitalists were eliminated from the economy as a function of the the fraction of capital κ initially owned by capitalists. Note that β^c is fixed by the choice of K/Y . However, β^l must be set for each value of κ according to

$$\beta^l = \frac{1}{\frac{1-\alpha}{1-\kappa^*} \frac{Y^*}{K^*} - 1}, \quad (94)$$

which is derived in Appendix A.

The capital stock always decreases by a greater percentage than κ .since capitalists always save more than laborers. The discount factor for laborers that would yield a capital-output ratio of 3.0 if there were no laborers is $\beta^l = 0.176 < \beta^c = 0.3$. (In annual terms, these would be 0.944 and 0.961 respectively.) If the capital-output ratio of 3.0 can only be achieved with capitalists, the laborers will be even less patient. Thus for all $\kappa \in [0, 1]$, $\beta^l < \beta^c$.

Output and laborer's utility, on the other hand, decrease by a smaller percentage than κ . For our baseline value of $\kappa = 0.2$, output would decrease by 10.6% and laborer's utility by the equivalent of 8.1% under this experiment. For Piketty's (2014) larger value of $\kappa = 0.3$, output would decrease by 16.3% and laborer's utility by 13.1%.

8 Political Economy

Now let us consider what happens when we introduce taxes into the model.

8.1 Price-Taking Capitalists

First suppose the capitalists are price-takers, so we know that in the steady state

$$\beta^c [1 + (1 - \tau^k)(F_K(K_{pt}^*, N_{pt}^*) - \delta)] = 1. \quad (95)$$

If we assume the worker has preferences (33) such that N_{pt}^* is fixed and

$$F(K, N) = N f\left(\frac{K}{N}\right),$$

then

$$\begin{aligned} (1 - \tau^k) \left(f' \left(\frac{K_{pt}^*}{N_{pt}} \right) - \delta \right) &= \frac{1}{\beta^c} - 1 \\ f' \left(\frac{K_{pt}^*}{N_{pt}} \right) &= \frac{\frac{1}{\beta^c} - 1}{1 - \tau^k} + \delta \\ K_{pt}^* &= N_{pt} (f')^{-1} \left(\frac{\frac{1}{\beta^c} - 1}{1 - \tau^k} + \delta \right) \end{aligned} \quad (96)$$

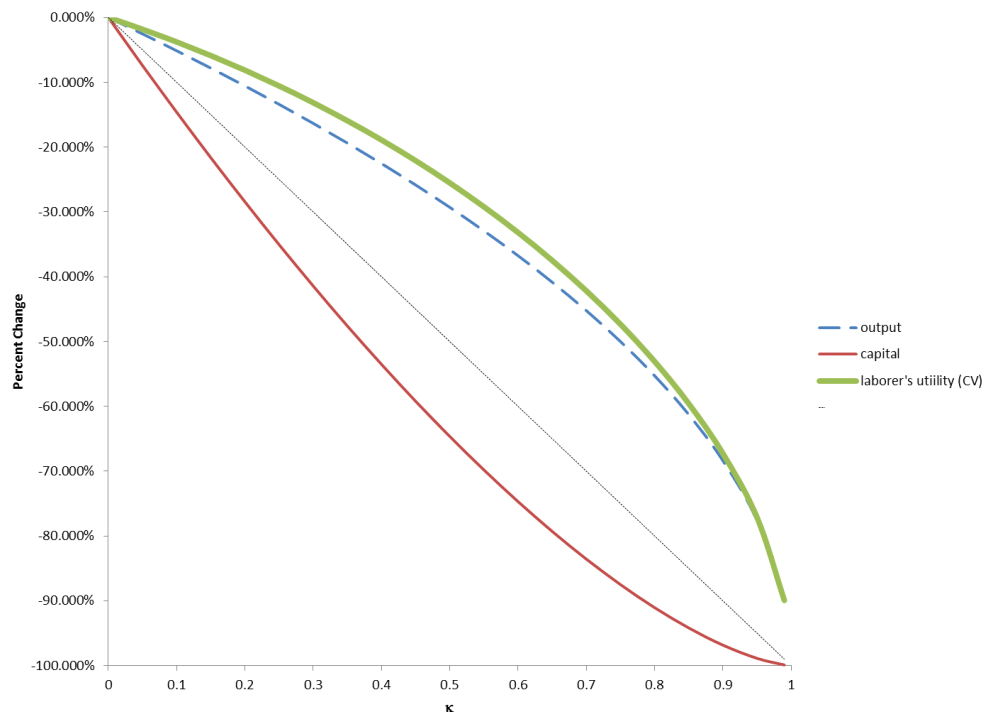


Figure 3: Percent change in output, capital, and laborer's utility (in terms of equivalent consumption) if price-taking capitalists are eliminated from the economy as a function of the fraction of capital κ initially owned by capitalists.

$$\begin{aligned}
y &= g(x) \\
\frac{dy}{dx} &= g'(x) \\
\frac{dx}{dy} &= \frac{1}{g'(g^{-1}(y))}
\end{aligned}$$

$$\frac{d^2x}{dy^2} = -\frac{g''(g^{-1}(y))}{(g'(g^{-1}(y)))^2} \frac{1}{g'(g^{-1}(y))} = -\frac{g''(g^{-1}(y))}{(g'(g^{-1}(y)))^3}$$

$$\frac{\partial K_{pt}^*}{\partial \tau^k} = -N_{pt} \frac{f''\left(\frac{K_{pt}^*}{N_{pt}}\right)}{\left(f'\left(\frac{K_{pt}^*}{N_{pt}}\right)\right)^3} \left(-\frac{\frac{1}{\beta^c} - 1}{(1 - \tau^k)^2}\right)$$

$$\frac{\partial K_{pt}^*}{\partial \tau^k} = N_{pt} \frac{f''\left(\frac{K_{pt}^*}{N_{pt}}\right)}{\left(\frac{\frac{1}{\beta^c} - 1}{1 - \tau^k} + \delta\right)^3} \left(-\frac{\frac{1}{\beta^c} - 1}{(1 - \tau^k)^2}\right)$$

$$\frac{\partial K_{pt}^*}{\partial \tau^k} = (1 - \tau_k) N_{pt} \frac{f''\left(\frac{K_{pt}^*}{N_{pt}}\right)}{\left(\frac{1}{\beta^c} - 1 + (1 - \tau_k)\delta\right)^3} \left(-\frac{1}{\beta^c} - 1\right)$$

Then

$$(k_{pt}^l)^* = (1 - \tau_l) sw(K_{pt}^*)$$

and

$$k_{pt}^* = \frac{K_{pt}^* - \mu (k_{pt}^l)^*}{1 - \mu} = \frac{K_{pt}^* - \mu(1 - \tau_l) sw(K_{pt}^*)}{1 - \mu}$$

In the case of price-taking, we find in the baseline calibration with $\gamma^c = \gamma^l = 1$, $\alpha = 1/3$, $\delta = 1.0$, $K/Y = 3$ (in annual terms), $\kappa = 0.2$, and $G/Y = 0.15$ that the standard result that capital taxes should be set to zero actually holds. Fig. 4 shows how welfare, measured as the change in equivalent consumption for the case when $\tau^k = 0.13$ (and capital taxes contribute 20% of tax revenue) varies for both capitalists and laborers as a function of the capital tax rate τ^k while setting τ^l to maintain the same G . The increase in the capital stock and wages that results from a lower tax on capital more than makes up for the increased tax on labor. However, laborers gain considerably less than capitalists. If capital taxes are only contributing 20% of tax revenue, the gain for laborers is equivalent to 0.4% of consumption. If they contribute 10%, the gain is only 0.1% of consumption. In contrast, if capital taxes contribute 20%, the gain for capitalists would be equivalent to 75% of their consumption.

However, the transition is not Pareto-improving. Some laborers experience a welfare loss equivalent to 5% of consumption as is shown in Fig. 2.

Deficit spending can be used to smooth out the loss of revenue so labor taxes do not have to be raised before wages start to rise. However, it is necessary for the government to build up some saving first. If the government simply

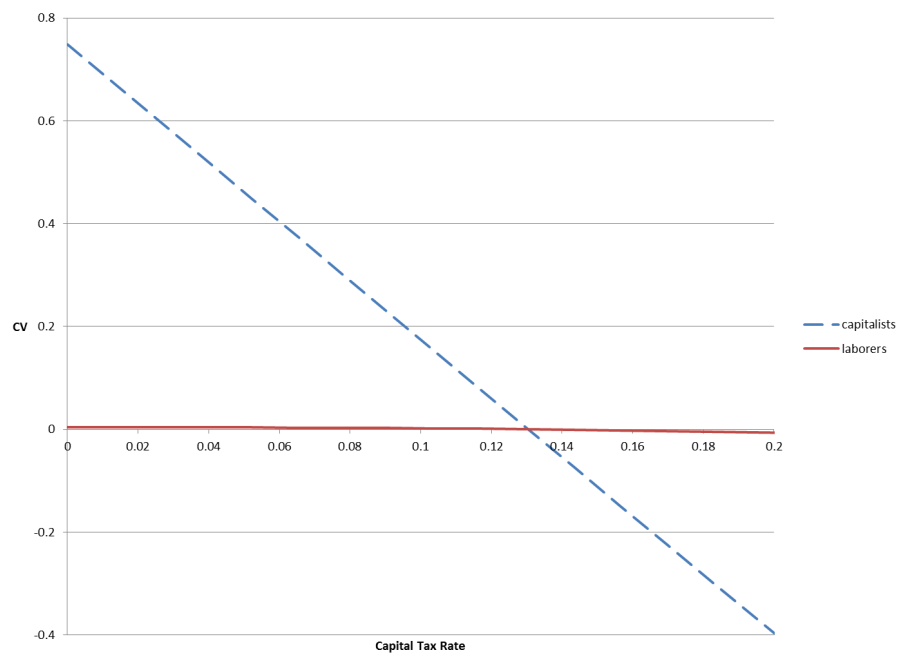


Figure 4: Compensating variation for capitalists and laborers as a function of the capital tax rate for the baseline calibration with taxes.

t	τ_k	τ_l	k	K	R	Δw_{at}	Δc_c	c_0^l	c_1^l	CV_l	ΔY
-2	0.13	0.179	0.765	0.0076	3.00	0.00%	0.0%	0.036	0.018	0.00%	0.00%
-1	0.13	0.179	0.765	0.0076	3.00	0.00%	0.0%	0.036	0.018	1.40%	0.00%
0	0.00	0.224	0.765	0.0076	3.30	-5.45%	10.0%	0.034	0.020	-3.91%	0.00%
1	0.00	0.226	0.842	0.0075	3.35	-6.41%	23.0%	0.034	0.019	-5.06%	-0.79%
2	0.00	0.224	0.941	0.0076	3.31	-5.67%	35.7%	0.034	0.019	-4.59%	-0.18%
3	0.00	0.222	1.039	0.0078	3.24	-4.41%	46.8%	0.034	0.019	-3.61%	0.85%
4	0.00	0.220	1.123	0.0081	3.18	-3.16%	55.4%	0.035	0.019	-2.59%	1.88%
5	0.00	0.218	1.189	0.0083	3.12	-2.11%	61.8%	0.035	0.019	-1.72%	2.75%
6	0.00	0.216	1.238	0.0085	3.08	-1.31%	66.3%	0.036	0.018	-1.05%	3.40%
7	0.00	0.216	1.273	0.0086	3.05	-0.73%	69.3%	0.036	0.018	-0.56%	3.88%
8	0.00	0.215	1.296	0.0087	3.04	-0.34%	71.3%	0.036	0.018	-0.23%	4.20%
9	0.00	0.214	1.311	0.0087	3.02	-0.07%	72.6%	0.036	0.018	0.00%	4.42%
10	0.00	0.214	1.321	0.0088	3.01	0.10%	73.5%	0.036	0.018	0.14%	4.56%
11	0.00	0.214	1.327	0.0088	3.01	0.21%	74.0%	0.036	0.018	0.24%	4.65%
12	0.00	0.214	1.331	0.0088	3.01	0.28%	74.3%	0.036	0.018	0.30%	4.71%
13	0.00	0.214	1.334	0.0088	3.00	0.33%	74.5%	0.036	0.018	0.34%	4.75%
14	0.00	0.214	1.335	0.0088	3.00	0.36%	74.6%	0.036	0.018	0.36%	4.77%
15	0.00	0.214	1.336	0.0088	3.00	0.37%	74.7%	0.036	0.018	0.38%	4.78%
20	0.00	0.214	1.338	0.0088	3.00	0.40%	74.8%	0.036	0.018	0.40%	4.80%
30	0.00	0.214	1.338	0.0088	3.00	0.40%	74.8%	0.036	0.018	0.40%	4.81%

Table 2: Transition for immediate elimination of capital taxes for baseline model with price-taking capitalists and a balanced budget.

t	τ_k	τ_l	D	k	K	R	Δw_{at}	Δc_c	c_0^l	c_1^l	CV_i	ΔY
-2	0.13	0.179	0.0000	0.765	0.0076	3.00	0.00%	0.0%	0.036	0.018	0.00%	0.00%
-1	0.13	0.179	0.0000	0.765	0.0076	3.00	0.00%	0.0%	0.036	0.018	1.40%	0.00%
0	0.00	0.203	0.0000	0.765	0.0076	3.30	-2.87%	10.0%	0.035	0.020	0.00%	0.00%
1	0.00	0.318	0.0011	0.842	0.0065	3.67	-21.06%	34.4%	0.028	0.022	-19.14%	-5.04%
2	0.00	0.232	0.0000	1.028	0.0069	3.54	-9.61%	58.6%	0.033	0.017	-8.70%	-3.41%
3	0.00	0.221	0.0000	1.214	0.0080	3.21	-3.85%	69.9%	0.035	0.018	-3.49%	1.32%
4	0.00	0.216	0.0000	1.300	0.0085	3.08	-1.22%	74.4%	0.036	0.018	-1.11%	3.47%
5	0.00	0.214	0.0000	1.334	0.0087	3.02	-0.11%	75.8%	0.036	0.018	-0.09%	4.39%
6	0.00	0.214	0.0000	1.345	0.0088	3.00	0.32%	76.0%	0.036	0.018	0.31%	4.74%
7	0.00	0.214	0.0000	1.347	0.0088	3.00	0.46%	75.8%	0.036	0.018	0.44%	4.86%
8	0.00	0.213	0.0000	1.346	0.0088	3.00	0.49%	75.6%	0.036	0.018	0.47%	4.88%
9	0.00	0.213	0.0000	1.344	0.0088	3.00	0.48%	75.4%	0.036	0.018	0.47%	4.87%
10	0.00	0.214	0.0000	1.342	0.0088	3.00	0.46%	75.2%	0.036	0.018	0.45%	4.86%
11	0.00	0.214	0.0000	1.341	0.0088	3.00	0.44%	75.1%	0.036	0.018	0.44%	4.84%
12	0.00	0.214	0.0000	1.340	0.0088	3.00	0.43%	75.0%	0.036	0.018	0.43%	4.83%
13	0.00	0.214	0.0000	1.339	0.0088	3.00	0.42%	74.9%	0.036	0.018	0.42%	4.82%
14	0.00	0.214	0.0000	1.339	0.0088	3.00	0.41%	74.9%	0.036	0.018	0.41%	4.82%
15	0.00	0.214	0.0000	1.338	0.0088	3.00	0.41%	74.9%	0.036	0.018	0.41%	4.81%
20	0.00	0.214	0.0000	1.338	0.0088	3.00	0.40%	74.8%	0.036	0.018	0.40%	4.81%

Table 3: Transition for immediate elimination of capital taxes for baseline model with price-taking capitalists and borrowing to defer increasing labor taxes for one period.

t	τ_k^c	τ_k^l	τ^l	D	k	K	R^c	Δw_{at}	Δc^c	CV_l	ΔY
-2	0.13	0.13	0.179	0.0000	0.765	0.0076	3.00	0.00%	0.0%	0.00%	0.00%
-1	0.13	0.13	0.179	0.0000	0.765	0.0076	3.00	0.00%	0.0%	0.00%	0.00%
0	0.17	0.13	0.186	0.0000	0.765	0.0076	2.92	-0.88%	-2.8%	0.00%	0.00%
1	0.00	0.00	0.199	-0.0005	0.744	0.0080	3.19	-0.77%	3.3%	0.00%	1.70%
2	0.00	0.00	0.200	-0.0005	0.791	0.0081	3.16	-0.56%	9.0%	0.00%	2.09%
3	0.00	0.00	0.192	-0.0006	0.834	0.0083	3.12	1.21%	13.3%	1.78%	2.81%
4	0.00	0.00	0.201	-0.0004	0.867	0.0083	3.12	0.14%	17.8%	0.76%	2.84%
5	0.00	0.00	0.204	-0.0004	0.901	0.0083	3.13	-0.41%	22.8%	0.18%	2.65%
6	0.00	0.00	0.204	-0.0003	0.940	0.0083	3.12	-0.39%	27.9%	0.12%	2.75%
7	0.00	0.00	0.205	-0.0003	0.979	0.0084	3.11	-0.24%	32.5%	0.20%	2.98%
8	0.00	0.00	0.205	-0.0003	1.014	0.0084	3.09	0.04%	36.6%	0.40%	3.27%
9	0.00	0.00	0.205	-0.0003	1.045	0.0085	3.08	0.32%	40.0%	0.61%	3.55%
10	0.00	0.00	0.205	-0.0003	1.071	0.0086	3.06	0.55%	42.8%	0.79%	3.78%
11	0.00	0.00	0.205	-0.0003	1.092	0.0086	3.05	0.72%	45.1%	0.92%	3.97%
12	0.00	0.00	0.205	-0.0003	1.110	0.0086	3.04	0.85%	47.1%	1.01%	4.12%
13	0.00	0.00	0.205	-0.0003	1.125	0.0087	3.03	0.93%	48.7%	1.07%	4.23%
14	0.00	0.00	0.205	-0.0002	1.138	0.0087	3.03	0.97%	50.2%	1.10%	4.31%
15	0.00	0.00	0.206	-0.0002	1.149	0.0087	3.03	1.00%	51.5%	1.11%	4.37%
20	0.00	0.00	0.207	-0.0002	1.192	0.0087	3.02	0.91%	56.7%	1.00%	4.50%
30	0.00	0.00	0.211	-0.0001	1.260	0.0087	3.02	0.48%	65.5%	0.55%	4.54%
40	0.00	0.00	0.214	0.0000	1.325	0.0088	3.01	0.20%	73.7%	0.23%	4.64%
50	0.00	0.00	0.214	0.0000	1.338	0.0088	3.00	0.40%	74.8%	0.40%	4.80%

Table 4: Pareto-improving transition resulting in elimination of capital taxes for baseline model with price-taking capitalists.

borrowers to defer the increase in labor taxes, laborers end up being much worse off as is shown in Table 3.

In order to eliminate the capital tax in a Pareto-improving fashion, it is necessary to raise taxes on capitalists before eliminating taxes. This transition is shown in Table 4.

9 Conclusion

A good chunk of what we thought we knew about macro—anything built around the assumption that we can model the macroeconomy as a price-taking general equilibrium—is going to be subject to revision in the light of these results. For example, the equity premium puzzle (Mehra and Prescott (1985)), which was already looking rather unproblematic with the rise of behavioral economics, may not be a puzzle at all if the biggest asset holders rationally deviate from the standard Euler equation. The same may hold for puzzles regarding international capital flows.

One puzzling feature of the model is that while reverting to price-taking behavior is not Pareto improving, Pareto-improving transitions from the price-

setting steady state do exist. Why do capitalists not promote them? I would speculate the answer is because they do not trust the government to get the transition right. Improving welfare for laborers is easy. Just increase the capital stock and compensate the initial old for the lost interest rate, and laborers will be better off. But this can only be accomplished by forcing capitalists to save more now. If the government knows the preferences of capitalists', capitalists can be rewarded to compensate them for their initial sacrifice, like with any profitable investment. But the government probably does not know their preferences that well. In particular, capitalists are likely more patient than economists usually assume. Also, there is a commitment problem. What guarantee is there that future generations will live up to the promise of compensating capitalists in the future? The government might just ask the capitalists to make additional sacrifices with promises of rewards in the farther future. The intrafamily bargaining problem will also complicate the situation since the present generation of capitalists is being asked to give up consumption now so that future generations are better off.

There are several avenues along which we would like to generalize the model. Ideally, instead of 2 types, capitalists and laborers we would have three types, capitalists, high-skilled labor, and low-skilled labor as is discussed in Feigenbaum (2018). The share of capital is fixed in the present stylized model since we have a Cobb-Douglas production function. With a KORV (2000) type production function, we could model the interaction between skill-biased technological change and price-setting behavior by capitalists. This may perhaps account for the rise in the share of capital documented by Piketty (2014).

Note that the model takes the inequality here as exogenous so it does not offer any insight into how this inequality might be reduced other than to suggest policies that would reduce the concentration of wealth among a small group of potential price-setters.

One of the shortcomings of efforts to employ the CKR framework as a growth model is its robust prediction that the capital-output ratio is dictated by the generational discount rate when we see such a wide variation in the capital-output ratio across countries. Tiny variations in the per annum discount rate amount to huge variations in the per generation discount rate. Effectively, the CKR explanation for variations in K/Y is that some cultures value their kids more than other cultures, which is problematic for a variety of reasons. In the segregated-economy model, the steady state value of K/Y is determined not by one preference parameter but instead by the distribution of several preference parameters across the population, which leads to a distribution of saving rates. Given that we observe a wide variation of saving rates both within and between countries, the segregated-economy model offers a much richer landscape of possible K/Y values even if we assume capitalists around the world all value their children the same.

What happens if some capitalists are price setters and some are price takers? The steady-state conditions for the two types would not be compatible if they have the same generational discount factor. The price setters would presumably disappear just as they would if the laborers are too patient. Likewise if the

price takers are more patient than the price setters. So the two types could only coexist if the price setters are sufficiently more patient than the price takers.

If capitalists do engage in price-setting behavior, how can policymakers combat this to restore efficiency to the economy? The most obvious solution would be for the government to invest in capital, though history makes one wary about going down this road, which can create more problems than it solves. Bruenig (2018) has proposed the creation of a social welfare fund that would be shared by all citizens that would pay dividends that serve as a universal basic income. Something similar already exists on a small scale in Alaska where some portion of oil revenues are shared by the citizenry. However, I do not wish to minimize the difficulty of generalizing such a fund to the scale that would be necessary to combat price-setting by capitalists so I will leave any detailed discussion of possible solutions to future work.

Another, possibly more politically feasible, alternative might be a progressive consumption tax as in Ræi (2018).

A Calibrating the Price-Taking Equilibrium with Log Utility for Capitalists

Suppose $\gamma_c = 1$. Then for the price-taking version of the segregated economy, we will have the simple policy rule

$$\begin{aligned} k_{t+1}^c &= \beta^c R(K_t) k_t^c \\ c_t^c &= (1 - \beta^c) R(K_t) k_t^c \\ R(K_t) k_t^c &= \frac{c_t^c}{1 - \beta^c} \end{aligned}$$

$$\begin{aligned} c_{t+1}^c &= (1 - \beta^c) R(K_{t+1}) k_{t+1}^c \\ &= (1 - \beta^c) R(K_{t+1}) \beta^c R(K_t) k_t^c \\ &= (1 - \beta^c) R(K_{t+1}) \beta^c \frac{c_t^c}{1 - \beta^c} \\ &= \beta^c R(K_{t+1}) c_t^c. \end{aligned}$$

$$K_{t+1} = \mu(1 - \tau_l) sw(K_t) + (1 - \mu) \beta^c R(K_t) k_t^c$$

In the steady state we must have $\beta^c R(K^*) = 1$

$$\begin{aligned} 1 + (1 - \tau_k) \left(\alpha \left(\frac{K^*}{N} \right)^{\alpha-1} - \delta \right) &= (\beta_c)^{-1} \\ \alpha \left(\frac{K^*}{N} \right)^{\alpha-1} - \delta &= \frac{(\beta_c)^{-1} - 1}{1 - \tau_k} \end{aligned}$$

$$\begin{aligned}
K^* &= N \left(\frac{(\beta_c)^{-1} - 1}{1 - \tau_k} + \delta \right)^{\frac{1}{\alpha-1}} \\
Y^* &= (K^*)^\alpha N^{1-\alpha} = N^\alpha \left(\frac{(\beta_c)^{-1} - 1}{1 - \tau_k} + \delta \right)^{\frac{\alpha}{\alpha-1}} N^{1-\alpha} = N \left(\frac{(\beta_c)^{-1} - 1}{1 - \tau_k} + \delta \right)^{\frac{\alpha}{\alpha-1}} \\
\frac{K^*}{Y^*} &= \frac{\left(\frac{(\beta_c)^{-1} - 1}{1 - \tau_k} + \delta \right)^{\frac{1}{\alpha-1}}}{\left(\frac{(\beta_c)^{-1} - 1}{1 - \tau_k} + \delta \right)^{\frac{\alpha}{\alpha-1}}} = \frac{\alpha}{\frac{(\beta_c)^{-1} - 1}{1 - \tau_k} + \delta} \\
\frac{(\beta_c)^{-1} - 1}{1 - \tau_k} + \delta &= \alpha \frac{Y^*}{K^*} \\
(\beta^c)^{-1} - 1 &= (1 - \tau_k) \left(\alpha \frac{Y^*}{K^*} - \delta \right) \\
\beta^c &= \frac{1}{1 + (1 - \tau_k) \left(\alpha \frac{Y^*}{K^*} - \delta \right)} \\
\kappa^* &= \frac{(1 - \mu) k_*^c}{K^*} \\
k_*^c &= \frac{K^* - \mu(1 - \tau_l) s w^*}{(1 - \mu) \beta^c R^*} \\
\kappa^* &= \frac{1 - \mu(1 - \tau_l) \frac{s w^*}{K^*}}{\beta^c R^*} = 1 - \mu(1 - \tau_l) \frac{s w^*}{K^*} \\
w^* &= (1 - \alpha) \left(\frac{K^*}{N} \right)^\alpha = (1 - \alpha) (K^*)^\alpha N^{-\alpha} \\
&= (1 - \alpha) \frac{Y^*}{N} \\
\kappa^* &= 1 - (1 - \tau_l) (1 - \alpha) \frac{\mu s}{N} \frac{Y^*}{K^*} \\
\frac{\mu s}{N} &= \frac{\mu \frac{\beta^l \eta}{1 + \beta^l \eta}}{\mu \frac{1 + \beta^l \eta}{1 + \beta^l \eta}} = \frac{\beta^l}{1 + \beta^l} \\
\kappa^* &= 1 - (1 - \tau_l) (1 - \alpha) \frac{\beta^l}{1 + \beta^l} \frac{Y^*}{K^*} \\
(1 - \tau_l) (1 - \alpha) \frac{\beta^l}{1 + \beta^l} \frac{Y^*}{K^*} &= 1 - \kappa^*
\end{aligned} \tag{97}$$

$$\frac{1 + \beta^l}{\beta^l} = \frac{(1 - \tau_l)(1 - \alpha) Y^*}{1 - \kappa^*} \frac{1}{K^*}$$

$$\beta^l = \frac{1}{\frac{(1 - \tau_l)(1 - \alpha) Y^*}{1 - \kappa^*} \frac{1}{K^*} - 1} \quad (98)$$

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