Deviations from Exponential Discounting and Present Bias in Continuous Time*

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Abstract

In a continuous-time lifecycle model with general time-inconsistent preferences, we establish necessary and sufficient conditions under which commitment to the initial plan will increase the realized objective function for all future selves. We also establish sufficient conditions under which the consumption profile over the lifecycle will be hump-shaped. We express these conditions in terms of what we call the future weighting factor, which measures the deviation of the discount function from an (arbitrarily chosen) exponential discount function. We find that a unanimous preference for committing to the initial path occurs when the future weighting factor at the **longest delay** is sufficiently large. Time-inconsistency is often characterized in terms of the concept of present bias, which relates here to the behavior of the marginal future weighting factor. Present bias is a necessary condition for the log consumption profile to be concave, but it is neither necessary nor sufficient for all the selves to prefer the initial path to the realized path.

JEL Classification: D11, D63, E21

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1 Introduction

With the advent of behavioral economics, many researchers have pursued the idea that a household really ought to be viewed as a multiplicity of selves. Here we consider a lifecycle environment where selves at different points in the lifecycle have inconsistent preferences and are naive about this inconsistency. We establish conditions on these preferences under which the selves would all be better off in hindsight if they committed to the consumption plan of the initial self. That is to say the initial consumption plan Pareto dominates the realized consumption plan determined at each period by the current self who acts independently and optimally based on her perception of the household problem.

In a standard dynamic economic problem, an individual chooses a plan of consumption over his remaining future subject to budget constraints. Typically, we assume he maximizes the utility that would be derived from the plan as he evaluates it at the start of his lifecycle. But if he is free to reconsider his plan at later dates, will he abide by the original plan or disobey it? Under the exponential discounting model of Samuelson (1937), all consumption plans chosen at different points over the lifetime will coincide with the initial plan. However, Samuelson admitted that he focused on this model because of the resulting mathematical simplicity rather than empirical evidence. Indeed, the model does a poor job of fitting people's choices of consumption. For example, graphs of average consumption over the lifecycle are generally hump-shaped, giving rise to what is often referred to as the "lifecycle consumption puzzle".[1]

To address the consumption puzzle, Strotz (1955) explored a variation of Samuelson's model with a different formulation of the discount function. A relative discount function discounts utility from future consumption based on the time delay until the individual enjoys this consumption rather than on the absolute time of the consumption, as Samuelson assumed.[2] Strotz (1955) noted that in his model an individual's marginal rate of substitution between consumption at different times will depend on when the individual is evaluating the utility from these consumptions. As a consequence, the optimal consumption plan can depend on when the plan is made. Strotz's main result is that, for any discount function other than Samuelson's exponential discount function, optimal plans made at different times will in fact not coincide over future intervals where both plans apply. Therefore, plans

[1] See Frederick et al. (2002), Rabin (2002), Carroll and Summers (1991), Attanasio and Weber (1995), Attanasio et al. (1999), among others for more details.

[2] See Cohen et al. (2020) for a review of various theoretical frameworks and empirical methods that have been employed to elicit intertemporal preferences.

for the future will be time-inconsistent. In a continuous time setup, Laibson (1997) later showed that hyperbolic discounting, which has become one of the more popular relative discount functions, could potentially account for several puzzles in the consumption and saving literature, including the consumption hump.

However, the empirical improvement that could be gained with time-inconsistent preference models did not erase the conceptual problems they introduce. Specifically, a rather discouraging aspect of a time-inconsistent discount function is the multiplicity of selves that inhabit the model, which in principle should complicate welfare analysis. While the different selves in Samuelson's exponential model will all agree about the best course of action, with time-inconsistent preferences the different selves will favor different courses of action.

To elaborate on this, consider the following example of how preferences may change over time, which is referred to as a preference reversal. Suppose that you face a choice between an extra \$1000 of consumption ten years from now or an extra \$1100 eleven years from now. With an exponential discounting function, your preference of whether to take the consumption early or with a premium later will remain the same as time passes and what was originally ten years in the future becomes the immediate present. However, with a nonexponential discount function, you might initially view the difference between ten years in the future and eleven years in the future as negligible and so you choose to take the \$1100. Ten years in the future, on the other hand, you might view \$1000 that you get immediately as more valuable than \$1100 in a year. Preferences that yield reversals like this where you prefer to move consumption forward as you approach the final decision point of an intertemporal tradeoff are said to be *present-biased*, or as Ericson and Laibson (2019) called it present-focused.[3] A hyperbolic discount function is one of the more prominent cases of a present-biased discount function. If, on the other hand, your preferences would lead to the opposite kind of reversal where you prefer to delay consumption as you approach the final decision point, they are said to be *future-biased*.

Having a model that is inhabited by multiple selves makes the basic problem of public choice among multiple people relevant even for one person, i.e. which self's preferences should be used to evaluate welfare? A common solution to the problem of evaluating welfare with time-inconsistent preferences is to use the preferences of the initial self. (See for example Laibson (1996), Laibson (1997), Laibson (1998), Laibson et al. (1998), O'Donoghue and Rabin (1999), O'Donoghue and Rabin (2000), O'Donoghue and Rabin (2001) among many

[3]Ericson and Laibson (2019) use the term present focus, rather than the more common term present bias, because they believe the word bias implies a prejudgment that the behavior is a mistake, which is not true in their view.

others.) Originally, this approach was justified by the thought that the most common time-inconsistencies observed empirically in households involve a present bias. Going back to the previous example, assuming the interest rate is less than 10%, \$1100 is worth more than \$1000, so the household would certainly be wealthier if it takes the \$1100. If it initially plans to take the \$1100, any failure to stick to that plan must be due to a lack of self control. Thus it is in the interest of society to help the household commit to the initial plan.

This argument may be valid for decisions about what to do over the next few hours or weeks, such as whether to go on a diet or to get up early. But it is not obvious whether this intuition is relevant for decisions over longer time scales. While a household will usually prefer consumption allocations that have a higher present value, this is not an exact rule. The intertemporal allocation of consumption across the lifecycle does matter for welfare. When deciding how much to consume now and how much to save for her retirement, can a twenty year old really understand the needs and concerns she will have when she is seventy? Gul and Pesendorfer (2004) point out that welfare analysis based on time-zero preferences "has the planner forever guarding the perceived interests of the nonexistent former selves," which they described as "odd". Likewise, Dewatripont et al. (2004) state that " there is no normative foundation" for equating welfare with time-zero preferences.[4] This is where our paper fits into the literature.

We take a novel approach by characterizing a general discount function in terms of a "future weighting factor" that measures the deviation of the discount function from a chosen exponential function. This representation of the discount function can accommodate all relative discount functions, and in particular it nests all the prominent forms of relative discount functions popular in the literature as special cases. We then find the necessary and sufficient condition for the initial consumption plan to Pareto dominate the consumption path determined at each period in terms of "future weighting factor". In other words, we are providing conditions under which the normative foundation missing in the literature does exist.

The paper closest to this one is Caliendo and Findley (2019), who considered a household in discrete time with quasihyperbolic preferences.[5] They compared the consumption plan determined by the time zero self to the realized consumption plan that results when each self knows that later selves will deviate from the earlier selves' plan if it is not to their liking.

[4] See Bernheim and Rangel (2009) for additional discussion.

[5] A quasihyperbolic discounting function is often used in discrete time as a simpler proxy for a hyperbolic discount function. This is like an exponential discount function for delays of one period or longer but with an additional discount factor between $t = 0$ and $t = 1$.

In the language of game theory, the realized plan must be a subgame perfect equilibrium in a game played by all of the selves.[6] They show that each of the different selves would actually obtain a higher value of their objective function if they all stuck to the time-zero plan than they get on the realized plan if the number of selves exceeds a certain threshold that turned out to be quite small in their setting. We generalize Caliendo and Findley (2019) by considering the welfare analysis problem in a continuous-time setting in which households are naive about the time-inconsistency of their preferences.

In order for the commitment path that results if the households all follow the plan of the initial self to Pareto dominate the realized consumption path, an infinite sequence of inequalities, one for each self, must be satisfied. In a companion paper, Feigenbaum and Raei (2021), we address the same issue in discrete time. Since there are only a finite number of delays to consider in discrete time, that setup provides the opportunity to isolate the effect of future weighting factors at every delay, which simplifies the interpretation of the results.[7] We find that the conditions for almost Pareto dominance[8] of the commitment allocation to the realized allocation can be expressed in terms of a lower bound on the future weighting factor at the longest delay. That is to say, the initial discount function has to fall sufficiently less slowly than an exponential at the end of life to get almost Pareto dominance of the commitment path. Likewise, the condition for almost Pareto dominance of the realized path is that the terminal future weight be lower than an upper bound.

The situation in continuous time is analogous but more complicated. We again obtain a lower bound on the future weighting factor at the end of life as the condition for the last self to prefer the commitment path, and to first order in the weights this lower bound is exactly analogous to its discrete-time counterpart. However, welfare at earlier ages is an integral rather than a sum in continuous time so at these ages we cannot isolate the effect of the terminal weight like we can in discrete time. However, to first order these integrals are also analogous to the corresponding sums in discrete time. In sum, instead of a sequence of lower bounds on the terminal weight, we obtain a continuous sequence of integral conditions that must all hold for Pareto dominance of the commitment path.

Thus far we have not discussed the choice of exponential used to compare the discount

[6] In the published version of Caliendo and Findley (2019), they considered sophisticated households. In earlier versions of the paper, they also considered the naive case like we do.

[7] This is in addition to the other obvious advantage of a discrete-time setup, which is that most economists are more comfortable with discrete time. On the other hand, it is harder to derive exact results in discrete time, so we have to employ linear approximations to obtain more tractable difference equations, whereas in the present continuous-time setup it is quite straightforward to derive exact results.

[8] To first order in the future weighting factors, utility at $t = 1$ has to be the same for both allocations.

function against when computing future weighting factors. This is of particular note because the sign of the future weighting factors plays an important role in the discrete-time paper. We say that a discount function exhibits *heavy future weighting* if the future weighting factors are all positive. Conversely, a discount function exhibits *light future weighting* if the weighting factors are all negative. Of course, the sign of the future weighting factors will depend on the choice of the exponential used to define the weighting factors. In discrete time, there is a natural choice for this exponential: the exponential function defined by the decay D_1 of the discount function after one discrete period. That is to say Feigenbaum and Raei (2021) exclusively use the exponential function $\exp(-\rho t) = D_1^t$. This choice of ρ yields a simple condition for present bias, that the future weighting factors are positive and increasing in the delay. Heavy future weighting becomes a necessary condition for a present bias, and, likewise, light future weighting is a necessary condition for a future bias.[9]

In continuous time, the connection between heavy future weighting and present bias breaks down. For example, a discount function with heavy future weighting can be future-biased in continuous time. This is in part a consequence of the fact that the choice of ρ is no longer crucial. We obtain exact results regarding the properties of the consumption path and how these depend on the future weighting factors, but these propositions do not depend on the choice of the exponential function. This is true even though a future weighting factor might be positive for some choice of ρ and negative for a different choice of ρ. It is the dynamics of the future weighting factors rather than their sign that matters for the properties we are investigating. This distinction is most apparent when considering the shape of the realized consumption function. The concepts of present and future bias are much more informative about the shape than heavy or light future weighting because, in continuous time, present and future bias impose conditions on the marginal future weighting factors, i.e. the rate of change of the future weighting factor.

As we did for the relative welfare of the realized and initial paths, we can also express the conditions for the lifecycle profile of log consumption along the realized path to be concave (or convex) in terms of conditions on the future weighting factors. As a reminder, having a concave log consumption profile is a sufficient condition for the consumption profile to be hump-shaped as we generally observe empirically.[10] Whereas the conditions for Pareto

[9]This result assumes the discount function is strictly positive and does not apply to myopic discount functions.

[10]Note that $\frac{d^2 \ln c(t)}{dt^2} = \frac{1}{c(t)}\frac{d^2 c(t)}{dt^2} - \left(\frac{d \ln c(t)}{dt}\right)^2$. A necessary condition for the consumption profile to be concave at t is that the log consumption profile is also concave at t. Since a hump-shaped consumption profile must be locally concave at its maximum, it follows that the log consumption profile must also be locally concave at the maximum.

dominance depend on the future weighting factors, the conditions for convexity and concavity of the log consumption profile depend on the marginal future weighting factors. Since the marginal future weighting factors are closely connected to present and future bias, we also get, with some caveats, that the conditions for concavity (convexity) imply that the discount function must be present-biased (future-biased).

While it is not obvious that the welfare properties of a discount function should be related to the shape of the corresponding consumption function, there is a connection. The exact condition for concavity also implies that terminal consumption will be higher on the commitment path than on the realized path, which is a necessary condition for the commitment path to Pareto dominate the realized path.[11]

This paper is organized as follows. Section 2 introduces the concept of a future weighting factor and describes the model environment in which we will explore the condition on the discount function. In section 3 we drive the condition on the discount function that makes commitment to the initial plan Pareto optimal. Section 4 describes the consumption profile. Section 5 explores the relationship between the Pareto conditions and concavity of the log consumption profile and finally section 6 provides concluding remarks.

2 Model environment

We focus on a lifecycle model in continuous time. A household lives with certainty to age T and receives an exogenous income $y(t)$ at age $t \in [0, T]$ which can either be consumed $c(t)$ or saved as assets $k(t)$.[12] We are assuming households are naive about the time-inconsistency of their preferences so they do not anticipate that they will reoptimize their consumption and saving plan in the future.

2.1 Discount function

Consider a household that lives in continuous time from 0 to T and at age $t \in [0, T]$ the household values future consumption allocations $c(s)$ for $s \geq t$ according to

$$U(t) = \int_t^T D(s-t)u(c(s))ds \tag{1}$$

[11] Richter (2020) focus on the curvature of the agents' time preferences and one of his results suggests that The First Welfare Theorem depends on convexity of agent's time inconsistency.

[12] We abstract from the labor supply decision.

for some utility function $u(c)$ and a discount function $D(t) \geq 0$. We normalize $D(0) = 1$ and assume $D(t) > 0$ on a neighborhood of 0 of positive measure.[13] Suppose that for a given $\rho > 0$, we define *future weighting factors* $\varepsilon(t) \geq -1$ for all $t \in [0, T]$ such that

$$D(t) = \exp(-\rho t)(1 + \varepsilon(t)). \tag{2}$$

Note that $\varepsilon(0) = 0$ by definition.

We say that a discount function $D(t)$ exhibits *heavy future weighting* if there is a $\rho > 0$ such that the $\varepsilon(t)$ defined by (2) are all nonnegative. Likewise, we say that a discount function exhibits *light future weighting* if there is a $\rho > 0$ such that the $\varepsilon(t)$ are all nonpositive.[14]

Note that if for $\rho > 0$,

$$\varepsilon_\rho(t) = \exp(\rho t)D(t) - 1 \geq 0 \tag{3}$$

for all $t \in [0, T]$ then the corresponding inequality would hold true for any $\rho' > \rho$. Thus in the case of heavy future weighting, it makes sense to define

$$\rho^* = \inf\{\rho > 0 : (\forall t \in [0, T])[D(t) \geq \exp(-\rho t)]\}. \tag{4}$$

Likewise, in the case of light future weighting, it makes sense to define

$$\rho^* = \sup\{\rho > 0 : (\forall t \in [0, T])[D(t) \leq \exp(-\rho t)]\}. \tag{5}$$

Example 1. *The hyperbolic discount function,*

$$D(t) = \frac{1}{1 + \eta t}, \tag{6}$$

where $\eta > 0$, is probably the most familiar nonexponential discount function. This is also an example of a discount function with heavy future weighting.

In this case we can show that $\rho^* = \eta$ and subsequently $\varepsilon(t)$ as follows.

We have

$$D(t) \geq \exp(-\eta t)$$

[13]This ensures none of the household's selves will want to consume all of its remaining wealth in the present instant.

[14]Note that there are discount functions that exhibit both heavy and light future weighting. For example, if $0 < \rho_1 < \rho_2$ and $D(t)$ satisfies $\exp(-\rho_2 t) < D(t) < \exp(-\rho_1 t)$, then $D(t)$ will exhibit heavy future weighting relative to $\exp(-\rho_2 t)$ and light future weighting relative to $\exp(-\rho_1 t)$.

since

$$\exp(\eta t) \geq 1 + \eta t, \tag{7}$$

with equality only if $t = 0$.[15]

Suppose that $\rho \in (0, \eta)$ *and*

$$\frac{1}{1+\eta t} \geq \exp(-\rho t)$$

for all $t \geq 0$. *Then*

$$\exp(\rho t) \geq 1 + \eta t \tag{8}$$

for all $t \geq 0$. *Let us define* $f(t;\rho)$ *as*

$$f(t;\rho) = \exp(\rho t) - \eta t - 1.$$

Then we have

$$f(0;\rho) = 0.$$

$$f'(t;\rho) = \rho \exp(\rho t) - \eta$$

$$f''(t;\rho) = \rho^2 \exp(\rho t) > 0.$$

We will have a minimum of f *at* t^* *that solves*

$$\exp(\rho t^*) = \frac{\eta}{\rho} > 1,$$

which is

$$t^* = \frac{1}{\rho} \ln\left(\frac{\eta}{\rho}\right) > 0. \tag{9}$$

Hence we have

$$\begin{aligned} f(t^*;\rho) &= \exp\left(\ln\left(\frac{\eta}{\rho}\right)\right) - \frac{\eta}{\rho}\ln\left(\frac{\eta}{\rho}\right) - 1 \\ &= \frac{\eta}{\rho}\left(1 - \ln\left(\frac{\eta}{\rho}\right)\right) - 1. \end{aligned}$$

Let $\delta = \frac{\eta}{\rho} - 1 > 0$, *so we can write* $f(t^*;\rho)$ *as*

$$f(t^*;\rho) = \delta - (1+\delta)\ln(1+\delta) \tag{10}$$

[15] This is a consequence of the strict convexity of the exponential function since $1 + \eta t$ is tangent to the function at $t = 0$.

Since[16]

$$\delta < (1+\delta)\ln(1+\delta) \tag{11}$$

for $\delta > 0$, we have $f(t^*;\rho) < 0$, which contradicts (8). Thus $\rho^* = \eta$. Therefore, for a hyperbolic discount function, a natural choice of the future weighting factor is

$$\varepsilon(t) = \frac{\exp(\eta t)}{1+\eta t} - 1 \geq 0 \tag{12}$$

with equality only for $t = 0$.

Somewhat counterintuitively, the choice of the unperturbed discount rate ρ that we will use to define the future weighting factors according to (3) will not actually matter for anything in what follows. Suppose we have a discount function $D(t)$ with heavy future weighting. For $\rho > \rho^*$, $D(t)$ will lie above $\exp(-\rho t)$ for all t whereas for $\rho < \rho^*$ there will be some s such that $D(s) < \exp(-\rho s)$. Thus the future weighting factor $\varepsilon_\rho(s)$ will be negative. Nevertheless our propositions are valid both for $\rho \geq \rho^*$ and $\rho < \rho^*$. The reverse story is true for discount functions with light future weighting.

A more familiar concept in the literature on time-inconsistent discount functions is *present bias*, and the hyperbolic discount function is a canonical example of present bias. While the sign of the future weighting factors depends on the choice of ρ, as we will see momentarily, ρ does not enter into the condition for present bias in continuous time.[17] Consequently, the only assumption we make about ρ in the propositions that we derive is that $\rho > 0$.

A present (future) bias occurs when a household will prefer a small (large) payoff now over a larger (smaller) payoff received after a delay $\Delta t > 0$, but that preference would be reversed if the comparison is made between a time t in the future and $t + \Delta t$. In appendix B we derive how this condition relates to future weighting factors.

As is the norm for economists, we will refer to $\varepsilon'(t)$ as the *marginal future weight* at t. However, if $\varepsilon(t) > -1$ we can also define

$$\mu(t) = \frac{\varepsilon'(t)}{1+\varepsilon(t)} \tag{13}$$

to be the *adjusted marginal future weight* at t. To first order in $\varepsilon(t)$, the marginal and adjusted marginal future weights are the same, but the correction in the denominator of

[16] See appendix A for the proof.

[17] In discrete time, Feigenbaum and Raei (2021) show that, if $\rho = -\ln D_1$, heavy future weighting is a necessary condition for a present bias, which in turn is a necessary condition for a concave log consumption profile.

(13) will be important for exact results.[18] In particular, a necessary condition for present-biased preference reversals at $t > 0$ to continue in the limit as $\Delta t \to 0$ is that

$$\mu(0) \leq \mu(t). \tag{14}$$

Since it is possible for this condition to hold even if $\varepsilon'(0) < 0$, so $\varepsilon(t) < 0$ for small $t > 0$,[19] present bias does not imply heavy future weighting or vice versa.

We will refer to a discount function that satisfies (14) with strict inequality for all $t > 0$ as *continuously present-biased*. Conversely, a discount function that satisfies

$$\mu(0) \geq \mu(t) \tag{15}$$

for all $t > 0$ is continuously future-biased. However, we must emphasize that this definition is only applicable if $\varepsilon(t) > -1$ for all t.[20]

Using (3) and (13), another expression involving $\mu(t)$ is that the instantaneous discount rate at the delay t is

$$\rho(t) = \frac{d \ln D(t)}{dt} = \rho - \frac{d \ln(1 + \varepsilon(t))}{dt} = \rho - \mu(t). \tag{16}$$

If we substitute this into (14)-(15), because it is constant ρ vanishes from the conditions for continuous present or future bias, which only depend on whether $\rho(t)$ is bigger or smaller than $\rho(0)$.[21] This accounts for why the choice of ρ is so unimportant in what follows. It is analogous to an integration constant, which matters for the numerical value of an antiderivative but cancels out of integrals. As we will see, what matters for the shape of the log consumption profile is the dynamics of how fast the discount function decays, and the

[18] In discrete time (Feigenbaum and Raei (2021)) with $\rho = -\ln D_1$, the exact analogous conditions for present and future bias only involve the marginal future weight without adjustment. This is why a present (future) bias implies heavy (light) future weighting in that case.

[19] Or even for all $t > 0$, as we will see in Section 4.

[20] For a myopic discount function that satisfies $\varepsilon(t) = -1$ for $t \geq t^*$, both present-biased and future-biased preference reversals can occur at different times. Assume $D(\Delta t) < 1$. For $t > t^*$, the household will be indifferent at time 0 between increasing consumption by $\Delta c > 0$ at t or $t + \Delta t$, but at time t it would certainly prefer to have that additional consumption immediately as opposed to Δt in the future. On the other hand, suppose $t < t^*$ is such that $\varepsilon(t) > -1$ and $t + \Delta t > t^*$, so $D(t) > 0$ and $D(t + \Delta t) = 0$. Then, as of time 0, the household would prefer to consume a positive amount at time t and nothing at $t + \Delta t$. Yet when the household reaches time t it would prefer to consume positive amounts both now and Δt in the future rather than only consuming now. Thus we have cases both where the household changes its preference to consume more in the present and where it prefers to consume more in the future.

[21] Prelec (2004) emphasizes a stronger condition that the instantaneous discount rate be increasing in t.

adjusted marginal future weight $\mu(t)$ captures these dynamics. Since ρ does not contribute to the dynamics, the choice of ρ is innocuous.

On the other hand, having the commitment path Pareto dominate the realized path will turn out to be associated with a large terminal future weight, not a present bias. Present-biased discount functions exist where commitment is not Pareto improving because the terminal future weight is not high enough relative to the other future weighting factors.

2.2 Household problem

Let us be more precise now about the problem that a household with the preferences (1) will solve. We assume the household earns an exogenous income stream $y(t) \geq 0$ for $t \in [0, T]$ with $y(t)$ strictly positive over some subset of positive measure and that savings at age t earn the instantaneous return $r(t)$. Thus the instantaneous budget constraint at age t is

$$\frac{dk(t)}{dt} = y(t) + r(t)k(t) - c(t). \tag{17}$$

with terminal condition

$$k(T) = 0 \tag{18}$$

Given $k(t)$, the household problem at age t can be described in the following way:

$$U(t) = \max_{c(s;t),k(s;t)} \int_t^T D(s-t)u(c(s;t))ds \tag{19}$$

s.t.

$$\frac{dk(s;t)}{ds} = y(s) + r(s)k(s;t) - c(s;t) \tag{20}$$

$$k(T;t) = 0. \tag{21}$$

For the remainder of the paper, we will assume that $u(c) = \ln(c)$.[22]

First, let us review some pertinent results from Feigenbaum (2016). We can define the gross return due to interest compounding between age 0 and t as

$$R(t) = \exp\left(\int_0^t r(s)ds\right) \tag{22}$$

[22]We assume here that households are naive about the time inconsistency of their preferences. With log utility, though, we would get the same results with more sophisticated households. See Marin-Solano and Navas (2009).

Note that

$$\frac{dR(t)}{dt} = r(t)\exp\Big(\int_0^t r(s)ds\Big) = r(t)R(t) \tag{23}$$

This means we can rewrite the budget constraint (17) using the gross return as

$$\frac{d}{dt}\Big(\frac{k(t)}{R(t)}\Big) = \frac{y(t)-c(t)}{R(t)}. \tag{24}$$

Integrating from t to T and using the terminal condition (18), we have

$$k(t) = \int_t^T \frac{R(t)}{R(s)}[c(s)-y(s)]ds. \tag{25}$$

This indicates that if we have the path of consumption and the exogenous income path, we can determine the saving $k(t)$ at any given age t.

If we define lifetime wealth as

$$W(t) = \int_t^T \frac{R(t)}{R(s)}y(s)ds + k(t), \tag{26}$$

this enables us to further rewrite the budget constraint as

$$\int_t^T \frac{R(t)}{R(s)}c(s)ds = W(t) \tag{27}$$

This equation encapsulates the familiar result that the present value of the consumption path from any given age t should be equal to the lifetime wealth at t.

Expressing the constraints in the form of (27), the Lagrangian of the household problem at t takes the following form:

$$\mathcal{L} = D(s-t)\ln(c(s;t)) - \lambda(t)\frac{R(t)}{R(s)}c(s;t)$$

The first order condition with respect to consumption is

$$\frac{\partial\mathcal{L}}{\partial c(s;t)} = \frac{D(s-t)}{c(s;t)} - \frac{\lambda(t)R(t)}{R(s)} = 0.$$

Hence

$$c(s;t) = \frac{1}{\lambda(t)}\frac{D(s-t)R(s)}{R(t)}$$

Inserting this into the lifetime budget constraint (27), we obtain

$$\frac{1}{\lambda(t)} \int_t^T D(s-t) ds = W(t),$$

and the household's age t plan consumption at age s is

$$c(s;t) = \frac{R(s)}{R(t)} \frac{D(s-t)}{\int_t^T D(s-t)ds} W(t). \tag{28}$$

But the household only follows this plan at $s = t$, for which

$$c(t) = \frac{1}{\int_t^T D(s-t)ds} W(t). \tag{29}$$

We define the marginal propensity to consume (MPC) out of total lifetime wealth at age t, including future income, as

$$m(t) = \frac{c(t)}{W(t)} = \frac{1}{\int_t^T D(s-t)ds} = \frac{1}{\int_0^{T-t} D(s)ds}. \tag{30}$$

The fact that the MPC depends on t indicates the fraction of wealth consumed varies over the life cycle.

Feigenbaum (2016) shows that a necessary condition for the solution to the household problem to satisfy the constraints (20) and (21) is

$$\lim_{t \to T} m(t)(T-t) = 1, \tag{31}$$

which implies that the household will consume all its remaining wealth in the last instant of life.

The growth rate of consumption is

$$G_c(t) \equiv \frac{d \ln c(t)}{dt} = \frac{d \ln m(t)}{dt} + \frac{d \ln W(t)}{dt}. \tag{32}$$

From (30), we have

$$\frac{d \ln m(t)}{dt} = \frac{D(T-t)}{\int_t^T D(s-t)ds} = m(t)D(T-t). \tag{33}$$

Using (26), we can write

$$\begin{aligned}\frac{dW(t)}{dt} &= \frac{dR(t)}{dt}\int_t^T \frac{y(s)}{R(s)}ds - \frac{R(t)}{R(t)}y(t) + \frac{dk(t)}{dt} \\ &= r(t)\left[\int_t^T \frac{R(t)}{R(s)}y(s)ds + k(t)\right] - c(t).\end{aligned}$$

Therefore, using (25) and (26),

$$\frac{dW(t)}{dt} = (r(t) - m(t))W(t),$$

or equivalently

$$\frac{d\ln W(t)}{dt} = r(t) - m(t). \tag{34}$$

Using (33), the growth rate of consumption is

$$G_c(t) = r(t) + m(t)[D(T-t) - 1].$$

By replacing $m(t)$ with (30), we have

$$\begin{aligned}G_c(t) &= r(t) + \frac{D(T-t) - 1}{\int_t^T D(s-t)ds} \\ &= r(t) + \frac{\int_t^T D'(s'-t)ds'}{\int_t^T D(s-t)ds}\end{aligned} \tag{35}$$

For an exponential discount function $D(t) = \exp(-\rho t)$, since $D'(t) = -\rho D(t)$, (35) simplifies to the familiar Euler equation with log utility

$$G_c(t) = r(t) - \rho. \tag{36}$$

With a nonexponential discount function, the deviation of the effective Euler equation from (36) will depend on

$$Z(t) = \frac{\int_t^T D'(s'-t)ds'}{\int_t^T D(s-t)ds} = \frac{D(T-t) - 1}{\int_t^T D(s-t)ds}. \tag{37}$$

The phenomena that we will study in the ensuing sections all stem from how the dynamics of the numerator $\int_t^T D'(s-t)ds$ differ from the dynamics of the denominator $\int_t^T D(s-t)ds$.

For an exponential discount function, since one integral is proportional to the other, the dynamics are exactly the same, so that is the least interesting case. More generally, the behavior of $Z(t)$ will depend on whether the discount function decays faster or slower than an exponential. The preferences of the different selves for the realized consumption path versus the consumption path of the initial self depend mostly on how the discount function compares to an exponential at the longest delays, which only matter for $Z(t)$ when $t \approx 0$. The shape of the consumption function depends on $Z(t)$ at all t and thus on how the discount function compares to an exponential at all delays.

We can understand this better if we introduce future weighting factors via (2). First, we can rewrite the MPC for this discount function as

$$m(t) = \frac{1}{\int_0^{T-t} \exp(-\rho s)[1+\varepsilon(s)]ds}. \tag{38}$$

Since for $\rho \neq 0$,

$$\int_0^t \exp(-\rho s)ds = \frac{1}{\rho}\left[1-\exp(-\rho t)\right], \tag{39}$$

we have

$$\begin{aligned} m(t) &= \frac{1}{\frac{1}{\rho}\left[1-\exp(-\rho(T-t))\right] + \int_0^{T-t}\exp(-\rho s)\varepsilon s ds} \\ &= \frac{\rho}{1-\exp(-\rho(T-t))}\left[1+\frac{\rho}{1-\exp(-\rho(T-t))}\int_0^{T-t}\exp(-\rho s)\varepsilon(s)ds\right]^{-1}. \end{aligned}$$

Approximating the MPC to first order in $\varepsilon(t)$, we can write it as

$$m(t) = \frac{\rho}{1-\exp(-\rho(T-t))}\left[1-\frac{\rho}{1-\exp(-\rho(T-t))}\int_0^{T-t}\exp(-\rho s)\varepsilon(s)ds\right] + O(\varepsilon^2). \tag{40}$$

We can see here that, for a given $W(t)$, heavy future weights will reduce $c(t)$ while light future weights will increase $c(t)$.

Replacing the discount function using (2), we can similarly rewrite $Z(t)$ as

$$Z(t) = \frac{\exp(-\rho(T-t))[1+\varepsilon(T-t)]-1}{\int_0^{T-t}\exp(-\rho s))[1+\varepsilon(s)]ds}. \tag{41}$$

This is equivalent to

$$\begin{aligned} Z(t) &= \frac{\exp(-\rho(T-t))[1+\varepsilon(T-t)]-1}{\frac{1-\exp(-\rho(T-t))}{\rho}+\int_0^{T-t}\exp(-\rho s)\varepsilon(s)ds} \\ &= -\frac{(1-\exp(-\rho(T-t)))\left[1-\frac{\exp(-\rho(T-t))}{1-\exp(-\rho(T-t))}\varepsilon(T-t)\right]}{\frac{1-\exp(-\rho(T-t))}{\rho}\left[1+\frac{\rho}{1-\exp(-\rho(T-t))}\int_0^{T-t}\exp(-\rho s)\varepsilon(s)ds\right]} \\ &= -\rho\frac{1-\frac{\exp(-\rho(T-t))}{1-\exp(-\rho(T-t))}\varepsilon(T-t)}{1+\frac{\rho}{1-\exp(-\rho(T-t))}\int_0^{T-t}\exp(-\rho s)\varepsilon(s)ds} \qquad (42) \end{aligned}$$

For small levels of $\varepsilon(t)$, we can rewrite (42) to first order in $\varepsilon(t)$ as

$$\begin{aligned} Z(t) &= -\rho\Bigg[1-\frac{\exp(-\rho(T-t))}{1-\exp(-\rho(T-t))}\varepsilon(T-t) \\ &\quad -\frac{\rho}{1-\exp(-\rho(T-t))}\int_0^{T-t}\exp(-\rho s)\varepsilon(s)ds\Bigg]+O(\varepsilon^2). \end{aligned}$$

Making use of (39), we can write this in the somewhat more convenient form

$$Z(t) = -\rho + \frac{\exp(-\rho(T-t))\varepsilon(T-t)+\rho\int_0^{T-t}\exp(-\rho z)\varepsilon(z)dz}{\int_0^{T-t}\exp(-\rho z')dz'} + O(\varepsilon^2).$$

Since

$$\frac{d}{dt}(-\exp(-\rho t)) = \rho\exp(-\rho t),$$

we can use an integration by parts

$$\begin{aligned} \rho\int_0^{T-t}\exp(-\rho z)\varepsilon(z)dz &= -\big[\exp(-\rho z)\varepsilon(z)\big]_0^{T-t}+\int_0^{T-t}\exp(-\rho z)\varepsilon'(z)dz \\ &= -\exp(-\rho(T-t))\varepsilon(T-t)+\int_0^{T-t}\exp(-\rho z)\varepsilon'(z)dz \qquad (43) \end{aligned}$$

to express $Z(t)$ as

$$Z(t) = -\rho + \frac{\int_0^{T-t}\exp(-\rho z)\varepsilon'(z)dz}{\int_0^{T-t}\exp(-\rho z')dz'} + O(\varepsilon^2) \qquad (44)$$

Thus, for a general discounting function with future weighting factor $\varepsilon(t)$, the effective

Euler equation in realized path is

$$G_c(t) = r(t) - \rho + \frac{\int_0^{T-t} \exp(-\rho z)\varepsilon'(z)dz}{\int_0^{T-t} \exp(-\rho z')dz'} + O(\varepsilon^2). \tag{45}$$

To zeroth order, i.e. when the discount function does not deviate from an exponential, we get back the familiar Euler equation (36). The first-order contribution to the consumption growth rate is then

$$G_c^1(t) = \frac{\int_0^{T-t} \exp(-\rho z)\varepsilon'(z)dz}{\int_0^{T-t} \exp(-\rho z')dz'}. \tag{46}$$

This is analogous to the result obtained by Feigenbaum and Raei (2021) in discrete time, where the first-order contribution to the effective Euler equation is a weighted average of the differences of the future weighting factors. It is the dynamics of the future weighting factors that cause a household with a nonexponential discount function to deviate from an exponential path of consumption. Moreover, changes of $\varepsilon(t)$ matter more for a delay t close to zero than for a delay close to T because the average (46) is weighted by the zeroth-order discount function $\exp(-\rho z)$.

3 Pareto condition

Now that we have specified the household problem with nonexponential discounting and described its solution, we can consider its policy implications. One of the main reasons for interest in present bias is the possibility that it may justify government interventions. Intuitively, if a household is continually readjusting its future plans to marginally increase its current consumption at the expense of a large decline in future consumption, one might expect the many selves to actually prefer that they could commit to the initial consumption plan for the entire lifecycle. If that is the case, then clearly there would be a legitimate reason for policymakers to help the household stay on that initial path–irrespective of how an external observer might weigh the opinions of the different selves. Conversely, a household with a future bias ought to continually readjust its plans to defer more consumption to the future, leading all selves to prefer the realized path over the commitment path except for the very first self, who must prefer the initial plan.

We will show, however, that it is not actually present or future bias that determines whether the commitment path Pareto dominates the realized path or vice versa. Instead, what matters is how the future weighting factors behave at the longest perceivable delay. We will now proceed to establish the condition on the discount function under which commitment to the initial plan will increase the realized objective function for all selves. We do our analysis in a steady state with constant interest rate $r(t) = r$.

3.1 Realized case

Since $C(t) = m(t)W(t)$, to determine the realized path of consumption we need to determine the realized path of total wealth. As we have seen, the dynamics of wealth are described by Eq. (34).

If we define

$$M(t) = \exp\left(\int_0^t m(s)ds\right), \tag{47}$$

then

$$\frac{dM(t)}{dt} = m(t)\exp\left(\int_0^t m(s)ds\right) = m(t)M(t),$$

and

$$\frac{d\ln M(t)}{dt} = m(t). \tag{48}$$

Inserting this, along with (22), into (34), we obtain

$$\frac{d\ln W(t)}{dt} = \frac{d\ln R(t)}{dt} - \frac{d\ln M(t)}{dt} = \frac{d}{dt}\ln\left(\frac{R(t)}{M(t)}\right).$$

Since $R(0) = M(0) = 1$, integrating this differential equation from 0 to t and exponentiating gives us

$$W(t) = \frac{R(t)}{M(t)}W(0). \tag{49}$$

Thus realized consumption at time t will be

$$c(t) = \frac{m(t)R(t)}{M(t)}W(0). \tag{50}$$

Writing out explicitly what $M(t)$ is in terms of the discount function,

$$c(t) = \frac{1}{\int_0^{T-t} D(s)ds}\exp\left(-\int_0^t \frac{ds}{\int_0^{T-s} D(s')ds'}\right)R(t)W(0) \tag{51}$$

Consequently, the realized utility as perceived by a household at age τ is

$$U^*(\tau) = \tag{52}$$
$$\int_\tau^T D(t-\tau) \ln \left[\frac{1}{\int_0^{T-t} D(s)ds} \exp\left(-\int_0^t \frac{ds}{\int_0^{T-s} D(s')ds'} \right) R(t)W(0) \right] dt.$$

3.2 Commitment case

If a household commits to the initial plan of the age-0 self, it will behave as though $D(t)$ is a time-consistent discount function. Therefore, on the commitment path the household will maximize

$$\int_0^T D(t) \ln c(t|0) dt$$

subject to

$$\int_0^T \frac{c(t|0)}{R(t)} dt = W(0).$$

The Lagrangian of the household problem is

$$\mathcal{L} = D(t) \ln c(t|0) - \lambda \frac{c(t|0)}{R(t)},$$

and the first order condition with respect to consumption is

$$\frac{\partial \mathcal{L}}{\partial c(t|0)} = \frac{D(t)}{c(t|0)} - \frac{\lambda}{R(t)} = 0.$$

Therefore, since $c(t|0) = c(0) = \lambda^{-1}$,

$$c(t|0) = D(t)R(t)c(0). \tag{53}$$

Inserting this into the budget constraint, we also have

$$c(0) \int_0^T D(t)dt = W(0).$$

Putting the solution for $c(0)$ back into (53), we obtain the initial plan for consumption

$$c(t|0) = \frac{D(t)R(t)W(0)}{\int_0^T D(s)ds} = \frac{R(t)W(0)}{\int_0^T \frac{D(s)}{D(t)} ds}. \tag{54}$$

Hence, utility from age t along the commitment consumption path is

$$U_c(\tau) = \int_\tau^T D(t-\tau) \ln \left[\frac{D(t)}{\int_t^T D(s)ds} \exp\left(-\int_0^t \frac{D(s)ds}{\int_s^T D(s')ds'} \right) R(t)W(0) \right] dt. \tag{55}$$

In appendix C, we show that this simplifies to

$$U_c(\tau) = \int_\tau^T D(t-\tau) \ln \left[\frac{D(t)}{\int_0^T D(s)ds} R(t)W(0) \right] dt. \tag{56}$$

3.3 Comparing the realized case with the commitment case

To compare the realized utility (52) to the commitment utility (55), we define the difference as

$$\Delta U(\tau) = U_c(\tau) - U^*(\tau). \tag{57}$$

If $\Delta U(\tau) > 0$, then committing to the time-zero consumption path will increase the realized utility for the household relative to the utility it would get from the consumption in the realized path. Note that if $\Delta U(\tau) \geq 0$ for all $\tau \in [0,T]$ with strict inequality in at least one case, the commitment path will Pareto dominate the realized path.

We can substitute (55) and (52) into (57) to obtain

$$\Delta U(\tau) = \int_\tau^T D(t-\tau) \ln \left[\frac{\frac{D(t)}{\int_0^T D(z')dz'}}{\frac{1}{\int_0^{T-t} D(z)dz} \exp\left(-\int_0^t \frac{ds}{\int_0^{T-s} D(s')ds'} \right)} \right] dt, \tag{58}$$

which can be simplified to

$$\Delta U(\tau) = \int_\tau^T D(t-\tau) \left[\ln \left(D(t) \frac{\int_0^{T-t} D(z)dz}{\int_0^T D(z')dz'} \right) + \int_0^t \frac{ds}{\int_0^{T-s} D(s')ds'} \right] dt. \tag{59}$$

For an exponential discount function, the realized path is equivalent to the commitment path. In other words, $\Delta U(\tau)$ must vanish if $\varepsilon(t) = 0$ for all $t \in [0,T]$. Thus $\Delta U(\tau) = O(\varepsilon)$ for all $\tau \in [0,T]$, and to first order $\Delta U(\tau)$ will be a linear function of the $\varepsilon(t)$, which we write

$$\Delta U(\tau) = \int_\tau^T B(t,\tau)\varepsilon(t)dt + O(\varepsilon^2). \tag{60}$$

In appendix D, we show that $B(z,\tau)$ is

$$\begin{aligned} B(z,\tau) &= \exp(-\rho(z-\tau))\Bigg[\Theta(z-\tau) \\ &+ \int_0^{T-z} \frac{\exp(-\rho t)\Theta(t-\tau)}{\int_0^{T-t}\exp(-\rho z')dz'}dt - \int_\tau^T \exp(-\rho t)M(t,z)dt\Bigg]. \end{aligned} \tag{61}$$

with

$$M(t,z) = \frac{1}{\int_0^T \exp(-\rho z')dz'} + \int_0^{\min\{t,T-z\}} \frac{ds}{\left(\int_0^{T-s}\exp(-\rho z')dz'\right)^2}, \tag{62}$$

and $\Theta(z)$ is the step function

$$\Theta(z) = \begin{cases} 0 & z<0 \\ 1 & z\geq 0 \end{cases}. \tag{63}$$

In the special case of $\tau=0$, we must also have $\Delta U(0)\geq 0$ since the commitment path is, by definition, the best possible path from the perspective of the $\tau=0$ self. If $\Delta U(0)$ had any first-order terms, the sign of $\Delta U(0)$ could be made positive by switching the sign of $\varepsilon(t)$, so we can infer that $\Delta U(0)=O(\varepsilon^2)$. In appendix E, we present a direct calculation that $B(t,0)=0$ for all $t\in[0,T]$.

We also show in appendix D that everything in brackets in (59) is at least first order in $\varepsilon(t)$, so we can rewrite that equation as

$$\Delta U(\tau) = \int_\tau^T \exp(-\rho(t-\tau))\left[\ln\left(D(t)\frac{\int_0^{T-t}D(z)dz}{\int_0^T D(z')dz'}\right) + \int_0^t \frac{ds}{\int_s^T D(s'-s)ds'}\right]dt + O(\varepsilon^2)$$

Therefore,

$$\begin{aligned} \frac{d}{d\tau}\Delta U(\tau) &= \rho\int_\tau^T \exp(-\rho(t-\tau))\left[\ln\left(D(t)\frac{\int_0^{T-t}D(z)dz}{\int_0^T D(z')dz'}\right) + \int_0^t \frac{ds}{\int_s^T D(s'-s)ds'}\right]dt \\ &\quad -\ln\left(D(\tau)\frac{\int_0^{T-\tau}D(z)dz}{\int_0^T D(z')dz'}\right) - \int_0^\tau \frac{ds}{\int_s^T D(s'-s)ds'} + O(\varepsilon^2) \\ &= \rho\Delta U(\tau) - \ln\left(D(\tau)\frac{\int_0^{T-\tau}D(z)dz}{\int_0^T D(z')dz'}\right) + \int_0^\tau \frac{ds}{\int_s^T D(s'-s)ds'} + O(\varepsilon^2) \end{aligned}$$

Hence,

$$\frac{d}{d\tau}\Delta U(0) = \rho\Delta U(0) - \ln\left(\frac{\int_0^T D(z)dz}{\int_0^T D(z')dz'}\right) + O(\varepsilon^2) = O(\varepsilon^2). \tag{64}$$

This is the continuous-time analog of the discrete-time result that $\Delta U_1 = O(\varepsilon^2)$ from Feigenbaum and Raei (2021).

In Feigenbaum and Raei (2021), we could take advantage of the fact that $\Delta U(\tau)$ is an ordinary sum in discrete time to use the $B(z,\tau)$ to derive a sequence of lower bounds for the terminal future weight such that $\Delta U(\tau) > 0$. It was then helpful to derive several properties of the $B(z,\tau)$. In continuous time, however, $\Delta U(\tau)$ is an integral, so we cannot isolate the effects of $\varepsilon(T)$. What we are able to do is to show that $B(T,\tau) > 0$ for all $\tau \in (0,T)$.

$$B(T,\tau) = \exp(-\rho T)\left[\exp(\rho\tau) - \int_\tau^T \exp(-\rho(t-\tau))M(t,T)\right]dt$$

Since

$$M(t,T) = \frac{1}{\int_0^T \exp(-\rho z')dz'},$$

we have

$$B(T,\tau) = \exp(-\rho T)\left[\exp(\rho\tau) - \frac{\int_0^{T-\tau}\exp(-\rho z)dz}{\int_0^T \exp(-\rho z)dz}\right] > 0$$

for $\tau > 0$.

As a practical matter, ignoring the set of measure zero of discount functions for which we can compute ΔU analytically, we will only be able to compute $\Delta U(\tau)$ numerically, approximating the integrals as sums on a grid with stepside $\Delta\tau > 0$. Since $\Delta U(T)$ is trivially zero, it is of no interest, and we have to deal with the terminal self separately. The largest $\tau < T$ that we will be able to compute $\Delta U(\tau)$ for will then be $T - \Delta\tau$. The only $\varepsilon(t)$ that will enter this calculation for $t > T - \Delta\tau$ will be $\varepsilon(T)$. The result that $B(T,\tau) > 0$ implies that we can, to first order, make $\Delta U(\tau)$ arbitrarily large for all $\tau \in (0, T-\Delta\tau]$, and thus positive, if we increase $\varepsilon(T)$ enough.

For the terminal self, we can do better than this in continuous time and obtain an exact result for a lower bound on $\varepsilon(T)$ such that the terminal self prefers the commitment path to the realized path, i.e. that $c(T|0) > c(T)$. Intuitively, one would expect the conditions for $c(T|0) > c(T)$ to be tighter than the conditions for $\Delta U(\tau) > 0$ for $\tau < T$ since what is optimal for the initial self will presumably become less optimal the older a self gets. In practice, if we make no restrictions whatsoever on the behavior of the future weighting factors, that does not have to be true. In discrete time, Feigenbaum and Raei (2021) established sufficient conditions under which it is true for small T up to 5.[23]

[23] These are also conditions necessary for the log consumption profile to be strictly concave, as we explore for continuous time in the next section.

The expression (51) for the realized consumption path derived in Section (3.1) is not defined in the limit as $t \to T$. In appendix F, we obtain an equivalent expression

$$c(t) = \frac{1}{\int_0^T D(s)ds} \exp\left(-\int_{T-t}^{T} \frac{(1-D(t'))}{\int_0^{t'} D(s')ds'}dt'\right) R(t)W(0).$$

Therefore, the terminal consumption in the realized path is

$$c(T) = \frac{1}{\int_0^T D(s)ds} \exp\left(-\int_{0}^{T} \frac{(1-D(t'))}{\int_0^{t'} D(s')ds'}dt'\right) R(T)W(0). \tag{65}$$

Meanwhile, the terminal consumption along the commitment path is by Eq. (54),

$$c(T|0) = \frac{D(T)R(T)W(0)}{\int_0^T D(s)ds} \tag{66}$$

Dividing (65) by (66), we obtain the ratio

$$\frac{c(T|0)}{c(T)} = D(T) \exp\left(\int_0^T \frac{(1-D(t'))\,dt'}{\int_0^{t'} D(s')ds'}\right).$$

Hence, the exact condition for $c(T|0) > c(T)$ is

$$D(T) > \exp\left(-\int_0^T \frac{(1-D(t))\,dt}{\int_0^t D(s)ds}\right). \tag{67}$$

We can rewrite (67) in terms of the future weighting factors as

$$\exp(-\rho T)(1+\varepsilon(T)) > \exp\left(-\int_0^T \frac{(1-\exp(-\rho t)(1+\varepsilon(t))}{\int_0^t \exp(-\rho s)(1+\varepsilon(s))ds}dt\right).$$

Using (39) in the second step, the argument of the exponent simplifies to

$$\begin{aligned}
\int_0^T \frac{(1-\exp(-\rho t)(1+\varepsilon(t))}{\int_0^t \exp(-\rho s)(1+\varepsilon(s))ds}dt &= \int_0^T \frac{(1-\exp(-\rho t))\left[1-\frac{\exp(-\rho t)}{1-\exp(-\rho t)}\varepsilon(t)\right]}{\int_0^t \exp(-\rho s)ds\left[1+\frac{\int_0^t \exp(-\rho s')\varepsilon(s')ds'}{\int_0^t \exp(-\rho s'')ds''}\right]}dt \\
&= \rho\int_0^T \frac{(1-\exp(-\rho t))\left[1-\frac{\exp(-\rho t)}{1-\exp(-\rho t)}\varepsilon(t)\right]}{(1-\exp(-\rho t))\left[1+\frac{\int_0^t \exp(-\rho s')\varepsilon(s')ds'}{\int_0^t \exp(-\rho s'')ds''}\right]}dt.
\end{aligned}$$

Thus the exact condition simplifies to

$$\exp(-\rho T)(1+\varepsilon(T)) > \exp\left(-\rho\int_0^T \frac{1-\frac{\exp(-\rho t)}{1-\exp(-\rho t)}\varepsilon(t)}{1+\frac{\int_0^t \exp(-\rho s)\varepsilon(s)ds}{\int_0^t \exp(-\rho s')ds'}}dt\right).$$

By rearranging this, we get

$$\varepsilon(T) > \exp\left(\rho\left[T-\int_0^T \frac{1-\frac{\exp(-\rho t)}{1-\exp(-\rho t)}\varepsilon(t)}{1+\frac{\int_0^t \exp(-\rho s)\varepsilon(s)ds}{\int_0^t \exp(-\rho s')ds'}}dt\right]\right)-1. \tag{68}$$

This establishes a lower bound on $\varepsilon(T)$ such that $c(T|0) \geq c(T)$ iff the terminal future weight is big enough relative to an aggregator of the other future weights. We emphasize here that so far we have not made any approximations. This is an exact necessary (but not sufficient) condition for the commitment consumption path to Pareto dominate the realized consumption path.

To get a better intuition of what this condition means, it is helpful, though, to consider what happens if we approximate the t integral in (68) to first order in the $\varepsilon(t)$.

$$\int_0^T \frac{1-\frac{\exp(-\rho t)}{1-\exp(-\rho t)}\varepsilon(t)}{1+\frac{\int_0^t \exp(-\rho s)\varepsilon(s)ds}{\int_0^t \exp(-\rho s')ds'}}dt = \int_0^T \left[1-\frac{\exp(-\rho t)}{1-\exp(-\rho t)}\varepsilon(t)-\frac{\int_0^t \exp(-\rho s)\varepsilon(s)ds}{\int_0^t \exp(-\rho s')ds'}\right]dt + O(\varepsilon^2).$$

Since the first term in this integral is T, the first-order condition for $c(T|0) > c(T)$ reduces to

$$\varepsilon(T) > \rho\int_0^T \left[\frac{\exp(-\rho t)}{1-\exp(-\rho t)}\varepsilon(t)+\frac{\int_0^t \exp(-\rho s)\varepsilon(s)ds}{\int_0^t \exp(-\rho s')ds'}\right]dt + O(\varepsilon^2). \tag{69}$$

Using (39) again, we can combine the fractions in the integrand to obtain

$$\varepsilon(T) > \int_0^T \left[\frac{\rho\int_0^t \exp(-\rho s)\varepsilon(s)ds+\exp(-\rho t)\varepsilon(t)}{\int_0^t \exp(-\rho s')ds'}\right]dt + O(\varepsilon^2).$$

An integration by parts

$$\begin{aligned}
\rho\int_0^t \exp(-\rho s)\varepsilon(s)ds &= -\int_0^t \frac{d}{ds}\left[\exp(-\rho s)\right]\varepsilon(s)ds \\
&= -\left[\exp(-\rho s)\varepsilon(s)\right]_0^t + \int_0^t \exp(-\rho s)\frac{d\varepsilon(s)}{ds}ds
\end{aligned}$$

yields the useful identity

$$\rho \int_0^t \exp(-\rho s)\varepsilon(s)ds = -\exp(-\rho t)\varepsilon(t) + \int_0^t \exp(-\rho s)\frac{d\varepsilon(s)}{ds}ds \tag{70}$$

Thus (69) simplifies to

$$\varepsilon(T) > \int_0^T \left[\frac{\int_0^t \exp(-\rho s)\frac{d\varepsilon(s)}{ds}ds}{\int_0^t \exp(-\rho s')ds'}\right] dt + O(\varepsilon^2) \tag{71}$$

Replacing integrals with sums and the derivative $\frac{d\varepsilon(s)}{ds}$ with the difference $\varepsilon(s+1)-\varepsilon(s)$, this condition is exactly analogous to the discrete-time condition developed in Feigenbaum and Raei (2021).[24]

We can understand the intuition behind this first-order result as follows. We have from (53) that

$$c(t|0) = (1+\varepsilon(t))\exp(-\rho t)R(t)c(0),$$

so

$$\ln c(t|0) = \varepsilon(t) - \rho + \ln(R(t)c(0)) + O(\varepsilon^2).$$

Thus $\varepsilon(t)$ measures the first-order deviation of the commitment path from the path of log consumption if the discount function was exactly $\exp(-\rho t)$. Meanwhile, the integrand on the right-hand side of (69) is the first-order contribution to $d\ln c(t)/dt$, $G_c^1(t)$, from (46). The inequality (71) compares $\varepsilon(T)$ to the integral $\int_0^T G_c^1(t)dt$ of this approximation to the growth rate of consumption over the lifecycle.

To close this section, we will present an example to illustrate that neither a present bias nor heavy future weighting are a sufficient cause to justify policy interventions to keep a household on its commitment path because there are cases of present-biased discount functions with heavy future weighting where some future selves would be hurt by such an intervention.

Suppose that the future weighting factors are

$$\varepsilon(t) = \eta\left(t + \frac{\exp(-\lambda t)-1}{\lambda}\right) \geq 0 \tag{72}$$

for $t \in [0,T]$, where $\eta, \lambda > 0$. Equality holds only for $t=0$. The inequality for $t>0$ follows

[24] See appendix G.

from

$$\varepsilon'(t) = \eta(1 - \exp(-\lambda t)) \geq \varepsilon'(0) = 0, \tag{73}$$

which also holds with equality only when $t = 0$. Since $\mu(t) = \frac{\varepsilon'(t)}{1+\varepsilon(t)}$ is also positive for $t > 0$, this is a present-biased discount function with heavy future weighting. Although we cannot obtain exact analytic results with this exponential future weighting factor, note that if λ is sufficiently large then $\varepsilon(t) \approx \eta$ for most t. Thus the upper bound of (71) will be approximately ηT whereas $\varepsilon(T) \approx \eta\left(T - \frac{1}{\lambda}\right)$, so we can see to first order that the possibility of $c(T) > c(T|0)$ exists with this future weighting factor.

Fig 1 shows the realized and commitment paths for log consumption in the particular case where $T = 60$, $\lambda = 1.0$, $\eta = 0.01$, and $\rho = r = 0.04$. To better see the deviation of the realized path from a straight line, we also provide a linear approximation to the realized path for comparison. With hyperbolic discounting, the realized consumption path would exceed the commitment path early in life, but then there is a reversal. The commitment path would Pareto dominate the realized path because the commitment path is higher for most of the lifecyle. This example follows that pattern for roughly the first half of the lifecycle, but in the later half the realized path is higher again. We compute that $c(T) = 0.0608 > c(T|0) = 0.0592$. Since the initial self will prefer the commitment path and the terminal self will prefer the realized path, neither path dominates. The selves will disagree which path is better, so there is no clear mandate for an intervention to keep the household on its initial plan.[25]

Keep in mind that, while we have been focusing on what happens at $\tau = T$, all of the $\Delta U(\tau)$ must be nonnegative for $\tau \in [0, T]$ in order for the commitment path to actually Pareto dominate the realized path. Consequently, the overall behavior of the derivative of the future weighting function determines whether the commitment path Pareto dominates the realized path, not just how future values compare to the value at $t = 0$.

4 Shape of the consumption profile

In this section, we study the shape of the consumption profile. Specifically, we focus on the curvature of the consumption plan and develop a necessary condition for the concavity (convexity) of the log consumption profile in terms of the future weighting factors we developed earlier, which, as we mentioned earlier, is the sufficient condition for the consumption

[25] For large λ, $\varepsilon'(t)$ is nearly constant with this future weighting function, so the results are very similar to what happens with a linear discount function, which we show in the next section will always be future-biased.

profile to be hump-shaped. As we will see, these conditions will impose restrictions on the future weighting factor at all delays rather than just on the weighting factor at the longest delay.

In the following, we will assume that the interest rate is a constant r since a dynamic interest rate will also affect the shape of the consumption profile. From (35), we have

$$\frac{d \ln c(t)}{dt} = r + Z(t),$$

where the term

$$Z(t) = \frac{\int_0^{T-t} D'(s')ds'}{\int_0^{T-t} D(s)ds} = \frac{D(T-t) - 1}{\int_0^{T-t} D(s)ds}$$

captures the effects of the discount function on the growth rate of consumption. It is the dynamics of $Z(t)$ that will determine the shape of the consumption profile in this environment. To have a concave (convex) log consumption profile at $t \in (0, T)$ we need $\frac{d^2 \ln c(t)}{dt^2} = Z'(t)$ to be nonpositive (nonnegative).

Eq. (41) expresses $Z(t)$ in terms of the future weighting factor. Let us define

$$Q(t) = \int_0^{T-t} D(s)ds = \int_0^{T-t} \exp(-\rho s)[1 + \varepsilon(s)]ds, \tag{74}$$

which is the denominator in $Z(t)$. Our assumption that $D(t)$ is positive on a neighborhood of $t = 0$ of positive measure guarantees that $Q(t) > 0$ for $t < T$.

Then we have

$$Q'(t) = -\exp(-\rho(T-t))[1 + \varepsilon(T-t)], \tag{75}$$

which we can use to derive the exact derivative of $Z(t)$:

$$\begin{aligned} Z'(t) &= \big[\exp(-\rho(T-t))\,[\rho(1+\varepsilon(T-t)) - \varepsilon'(T-t)]\,Q(t) \\ &+ (\exp(-\rho(T-t))[1+\varepsilon(T-t)] - 1)\exp(-\rho(T-t))[1+\varepsilon(T-t)]\big]\frac{1}{Q(t)^2}. \end{aligned}$$

We can rearrange this to obtain

$$\begin{aligned} Z'(t) &= \frac{\exp(-\rho(T-t))}{Q(t)}\Bigg[-\varepsilon'(T-t) \\ &+ \frac{(\exp(-\rho(T-t))[1+\varepsilon(T-t)] - 1) + \rho Q(t)}{Q(t)}[1+\varepsilon(T-t)]\Bigg]. \end{aligned} \tag{76}$$

Using (74), the numerator in the second term of (76) is

$$\exp(-\rho(T-t))\varepsilon(T-t)+\rho\int_0^{T-t}\exp(-\rho s)\varepsilon(s)ds=\int_0^{T-t}\exp(-\rho s)\varepsilon'(s)ds,$$

according to the identity (70). Thus the exact derivative simplifies to

$$Z'(t)=-\frac{\exp(-\rho(T-t))}{Q(t)}\left[\varepsilon'(T-t)-\frac{\int_0^{T-t}\exp(-\rho s)\varepsilon'(s)ds}{Q(t)}[1+\varepsilon(T-t)]\right]. \tag{77}$$

To get some intuition about this result, let us consider what happens when deviations from exponential discounting are small by approximating this derivative to first order in the future weighting factors. Everything in square brackets is already to first order in $\varepsilon(t)$, so the other factors of $Q(t)$ simplify to $\int_0^{T-t}\exp(-\rho z)dz$, and we disregard the extra factor of $\varepsilon(T-t)$. Thus (77) reduces to

$$Z'(t)=-\frac{\exp(-\rho(T-t))}{\int_0^{T-t}\exp(-\rho z')dz'}\left[\varepsilon'(T-t)-\frac{\int_0^{T-t}\exp(-\rho z)\varepsilon'(z)dz}{\int_0^{T-t}\exp(-\rho z')dz'}\right]+O(\varepsilon^2).$$

Because the factors outside the square brackets are unambiguously negative, the log consumption profile will be concave at $T-t$ to first order iff

$$\varepsilon'(t)\geq\frac{\int_0^t\exp(-\rho z)\varepsilon'(z)dz}{\int_0^t\exp(-\rho z')dz'}, \tag{78}$$

and the profile will be strictly concave if the inequality is strict. That is to say, the log consumption profile will be concave if the marginal future weight at t is bigger than a weighted average of the marginal future weights from 0 to t. This is exactly analogous to the result obtained in discrete time by Feigenbaum and Raei (2021). However, in continuous time we can show exactly that it is not really the marginal future weights but the adjusted marginal future weights (if they are defined) that matter for determining the shape of the log consumption profile. However, adjusted marginal future weights are only defined if $\varepsilon(t)>-1$. First let us consider the more general case.

Example 2. *Myopia: Suppose that we have a myopic discount function such that $\varepsilon(t)=-1$ for $t>t^*$ and $\varepsilon(t)>-1$ for $t\in(t^*-\delta,t^*)$ for some $\delta>0$. In this case, (77) gives us*

$$\frac{d^2\ln c(t)}{dt^2}=Z'(t)=0$$

for $t < T-t^*$ since $\varepsilon(T-t) = -1$ and $\varepsilon'(T-t) = 0$. As long as the household's effective time horizon is shorter than its actual time horizon, the log consumption profile will be linear.

Although it is possible to construct examples where $D(t)$ and $\varepsilon(t)$ are differentiable at t^*, it is easier to envision cases where $\varepsilon(t)$ and $D(t)$ are not differentiable at t^*, but they are differentiable in a deleted neighborhood of t^*. Suppose the left derivative of $D(t)$ at t^* is $-\nu < 0$, and the right derivative is 0.

Define $z^* = T - t^*$, which will be the age at which the kink in the consumption profile caused by the myopia happens. Thus if $z < z^*$, $T - z > t^*$, and we have from (37) that

$$Z(z) = \frac{-1}{\int_0^{T-z} D(s)ds} = \frac{-1}{\int_0^{t^*} D(s)ds} = Z(z^*). \tag{79}$$

For $z > z^*$, (37) gives

$$Z(z) = \frac{D(T-z)-1}{\int_0^{T-z} D(s)ds} = \frac{D(T-z)-1}{\int_0^{t^*} D(s)ds - \int_{T-z}^{t^*} D(s)ds}.$$

. Thus

$$\begin{aligned} Z(z) - Z(z^*) &= \frac{D(T-z)-1}{\int_0^{t^*} D(s)ds - \int_{T-z}^{t^*} D(s)ds} - \frac{-1}{\int_0^{t^*} D(s)ds} \\ &= \frac{[D(T-z)-1]\int_0^{t^*} D(s)ds + \int_0^{t^*} D(s)ds - \int_{T-z}^{t^*} D(s)ds}{\int_0^{t^*} D(s')ds' \int_0^{T-z} D(s'')ds''} \\ &= \frac{D(T-z)\int_0^{t^*} D(s)ds - \int_{T-z}^{t^*} D(s)ds}{\int_0^{t^*} D(s')ds' \int_0^{T-z} D(s'')ds''}. \end{aligned}$$

Dividing both sides by $z - z^*$ and taking the left-hand limit of $z \to z^*$, we get

$$\lim_{z \downarrow z^*} \frac{Z(z) - Z(z^*)}{z - z^*} = \frac{\nu}{\int_0^{t^*} D(s)ds} > 0.$$

The slope of the log consumption profile will kink upward at z^* in this case, so the profile will be strictly convex in a neighborhood of z^*. Assuming the discount function is strictly positive at shorter delays than t^*, the shape of the log consumption profile later in the lifecycle can be addressed with the ensuing results that assume $\varepsilon(t) > -1$.

Without making any approximations, we get from (77) that the log consumption profile is concave at $T - t$ iff

$$\varepsilon'(t) \geq \frac{\int_0^t \exp(-\rho s)\varepsilon'(s)ds}{\int_0^t \exp(-\rho s')(1+\varepsilon(s'))ds'}(1+\varepsilon(t)). \tag{80}$$

If the inequality is strict then the concavity will also be strict.

Let us suppose the log consumption profile is concave for all $t \in [0, T]$, so (80) holds everywhere. By definition we have $\varepsilon(0) = 0$. However, much depends on $\varepsilon'(0)$. If $\varepsilon'(0) > 0$, we must have $\varepsilon(t) > 0$ for t in a positive neighborhood of zero. Using continuous induction, if $\varepsilon(s) > 0$ for $s \in [0, t]$ and $\varepsilon'(s) > 0$ for $s \in [0, t)$, the right-hand side of (80) will be positive, so $\varepsilon'(t) > 0$, and $\varepsilon(s) > 0$ in a neighborhood of t greater than t. Thus $\varepsilon(t) > 0$ for all $t \in (0, T]$, and $\varepsilon'(t) > 0$ for all $t \in [0, T]$.

Proposition 3. *If $\varepsilon(t)$ is strictly increasing for $t > 0$ in a neighborhood of 0, a necessary condition for the log consumption profile to be strictly concave is that the discount function exhibit heavy future weighting with weights $\varepsilon(t)$ that are strictly increasing in the delay t. Conversely, if $\varepsilon(t)$ is strictly decreasing for $t > 0$ in a neighborhood of 0, a necessary condition for the log consumption profile to be strictly convex is that the discount function exhibit light future weighting with weights $\varepsilon(t)$ that are strictly decreasing in the delay t.*

Regarding the caveat about the behavior of $\varepsilon(t)$ for t near zero, the issue is that

$$\lim_{t\to 0} \frac{\int_0^t \exp(-\rho s)\varepsilon'(s)ds}{\int_0^t \exp(-\rho s')[1+\varepsilon(s')]ds'}$$

need not be zero even though the limit of the numerator is zero. By l'Hopital's rule,

$$\lim_{t\to 0} \frac{\int_0^t \exp(-\rho s)\varepsilon'(s)ds}{\int_0^t \exp(-\rho s')[1+\varepsilon(s')]ds'} = \lim_{t\to 0} \frac{\exp(-\rho t)\varepsilon'(t)}{\exp(-\rho t)[1+\varepsilon(0)]} = \varepsilon'(0).$$

Thus (80) will always hold with equality at $t = 0$. We get a difference between continuous time and discrete time here because the analog to the right-hand side of (80) at $t = 1$ in Feigenbaum and Raei (2021) would depend on $\Delta\varepsilon_0 = \varepsilon_1 - \varepsilon_0$, which by definition was 0. However, $\varepsilon'(0)$ does not have to be zero in continuous time.

Example 4. *Linear $\varepsilon(t)$: As an example of why the caveat is necessary, suppose that $\varepsilon(t) =$*

mt. Inserting this linear future weighting factor into (41), we get

$$Z(t) = \frac{\exp(-\rho(T-t))[1+m(T-t)] - 1}{\int_0^{T-t} \exp(-\rho s)[1+ms]ds}. \tag{81}$$

Doing the integral in the denominator,[26] *this simplifies to*

$$Z(t) = -(\rho)^2 \frac{1}{\rho + m\frac{1}{1-m(T-t)\frac{\exp(-\rho(T-t))}{1-\exp(-\rho(T-t))}}}, \tag{82}$$

which reduces to the usual $-\rho$ *in the exponential case where* $m = 0$.

Let us define

$$\chi(t) = -m(T-t)\frac{\exp(-\rho(T-t))}{1-\exp(-\rho(T-t))}, \tag{83}$$

so we can rewrite

$$Z(t) = -(\rho)^2 \frac{1}{\rho + m\frac{1}{1+\chi(t)}}. \tag{84}$$

Using this we have

$$\begin{aligned} Z'(t) &= (\rho)^2 \frac{1}{\left(\rho + m\frac{1}{1+\chi(t)}\right)^2} \left[-m\frac{1}{(1+\chi(t))^2}\right] \chi'(t) \\ &= \frac{-m(\rho)^2}{(m+\rho(1+\chi(t)))^2}\chi'(t). \end{aligned}$$

Since we can write $\chi(t)$ *as*

$$\chi(t) = m(T-t)\left[1 - \frac{1}{1-\exp(-\rho(T-t))}\right],$$

then, as we show in appendix I, we have

$$\chi'(t) = -m\exp(-\rho(T-t))\frac{\exp(-\rho(T-t)) - 1 + \rho(T-t)}{(1-\exp(-\rho(T-t)))^2}. \tag{85}$$

This means

$$Z'(t) = \frac{m^2(\rho)^2}{(m+\rho(1+\chi(t)))^2}\exp(-\rho(T-t))\frac{\exp(-\rho(T-t)) - 1 + \rho(T-t)}{(1-\exp(-\rho(T-t)))^2} \geq 0.$$

[26] See appendix H for details.

The inequality follows by applying (7), and equality holds only if $t = T$. Thus, with linear $\varepsilon(t)$, the log consumption profile is always strictly convex.

Note that the first-order convexity condition (78) always holds with equality since $\varepsilon'(z) = m$ factors out of the right-hand side. In appendix J we show that the exact condition (80) holds strictly for $m \neq 0$.

This is one of the few results in continuous time that differs markedly from what Feigenbaum and Raei (2021) obtained in discrete time. In discrete time, there are cases where a linear ε_s can yield a strictly concave log consumption profile. However, there is a key difference that likely accounts for this discrepancy. In discrete time, we can fix the exponential factor of the discount function based on the discount function at a delay of one time period. Thus the future discount factors do not deviate from zero until we get to a delay of two periods. In continuous time, $\varepsilon(s)$ can deviate from zero for any positive delay. We conjecture that with a discontinuous that jumps to a linear function after some delay $\tau > 0$, we might be able to get strict concavity except for $t > T - \tau$, where we would have only weak concavity.

Example 5. *Quadratic case: As we show in appendix K, a quadratic*

$$\varepsilon(t) = mt + \frac{1}{2}bt^2,$$

will satisfy the first-order strict concavity condition, as is also true in discrete time. However, we also show that $\varepsilon(t)$ need not solve the exact condition for convexity if $bT > \rho$. A quadratic $\varepsilon(t)$ can quickly grow to be much larger than 1 for large t, resulting in a nonmonotonic discount function. The assumption that we can ignore second- and higher-order terms in the concavity condition amounts to assuming b is small enough that the discount function does not deviate too much from an exponential function.

At the other extreme, if $m \neq 0$, for t small relative to $\frac{|m|}{b}$, the preceding argument regarding linear $\varepsilon(t)$, invoking the exact rather than the first-order condition, will apply, and the log consumption profile will be convex. Thus, a key feature necessary to get a concave log consumption profile will be that $\varepsilon'(0) = 0$, which is not satisfied if we have a nonzero linear term.

While adjusted future weights are not always defined for all t, as in the myopic example, we can shed light on these results for linear and quadratic $\varepsilon(t)$ if we rewrite (80) in terms of the adjusted future weighting factor $\mu(t)$. This also facilitates interpretation of the condition for concavity in terms of present and future bias. If $\varepsilon(t) > -1$ for all $t \in [0, T]$, we can divide

(80) by $1+\varepsilon(t)$. Thus we have that the log consumption profile is concave at $T-t$ iff

$$\mu(t) = \frac{\varepsilon'(t)}{1+\varepsilon(t)} \geq \frac{\int_0^t \exp(-\rho s)\varepsilon'(s)ds}{\int_0^t \exp(-\rho s')[1+\varepsilon(s')]ds'} = \frac{\int_0^t D(s)\mu(s)ds}{\int_0^t D(s')ds'}, \tag{86}$$

where we employ (2) to get the final equality. Strict concavity (convexity) at $T-t$ results if the adjusted marginal future weight is larger (smaller) than a weighted average of the adjusted marginal future weights at shorter delays than t, where the weights are the exact discount function. This is analogous to the first-order concavity condition (78) without adjustment to a weighted average of the marginal future weights at shorter delays. However, this result in terms of the adjusted marginal future weights is exact.

Recall that (14)-(15) respectively express the condition for present and future bias regarding shifts in consumption by an infinitesimal delay in terms of the adjusted marginal future weights. Combining these definitions with the condition (86) (or the reverse in the case of convexity) yields a proposition analogous to Proposition 3. The new proposition is also cleaner since we do not need to make any additional assumptions about the behavior of μ at 0 like we did in Proposition 3. Since (14) is a weaker condition than (86), a present bias is a necessary but not sufficient condition for a concave log consumption profile.

Proposition 6. *If $\varepsilon(t) > -1$ and satisfies (80) (or equivalently (86)) for all $t \in (0, T]$, so the log consumption profile is concave, then the discount function will be continuously present biased. Conversely, if the log consumption profile is convex and $\varepsilon(t) > -1$ for all $t \in (0, T]$, the discount function will be continuously future biased.*

The proof again follows by induction. If $\mu(s) \geq \mu(0)$ for all $s \in [0, t)$,

$$\mu(t) \geq \frac{\int_0^t D(s)\mu(s)ds}{\int_0^t D(s')ds'} \geq \mu(0),$$

and similarly if the log consumption profile is convex.

This proposition clarifies why the log consumption profile is always convex with a linear future weighting factor. If $\varepsilon(t) = mt$, then

$$\mu(t) = \frac{m}{1+mt} \leq m = \mu(0)$$

with equality only if $m = 0$. Thus the discount function with a linear future weighting factor is always continuously future-biased.

Note that if $\mu(t)$ is increasing,[27] we will have

$$\mu(t) = \frac{\int_0^t D(s)\mu(t)ds}{\int_0^t D(s')ds'} \geq \frac{\int_0^t D(s)\mu(s)ds}{\int_0^t D(s')ds'}.$$

Likewise if $\mu(t)$ is strictly increasing, we will have

$$\mu(t) = \frac{\int_0^t D(s)\mu(t)ds}{\int_0^t D(s')ds'} > \frac{\int_0^t D(s)\mu(s)ds}{\int_0^t D(s')ds'}.$$

Thus we also have a sufficient but not necessary condition for the log consumption profile to be (strictly) convex or concave.

Proposition 7. *If $\varepsilon(t) > -1$ and $\mu(t)$ is increasing, then (80) (or equivalently (86)) will be satisfied for all t, and the log consumption profile will be concave. If $\mu(t)$ is strictly increasing, these inequalities will be satisfied strictly, and the log consumption profile will be strictly concave. Conversely, if $\mu(t)$ is decreasing, the log consumption profile will be convex. If $\mu(t)$ is strictly decreasing, the log consumption profile will be strictly convex.*

This proposition clarifies our result for a quadratic $\varepsilon(t) = \frac{1}{2}bt^2$. Since we have

$$\mu(t) = \frac{bt}{1 + \frac{1}{2}bt^2} \geq 0 = \mu(0),$$

this discount function is continuously present-biased. However, $\mu(t)$ is nonmonotonic, increasing for small t and approaching $\frac{1}{2t}$ for large t. Since $\mu(t) > 0$ almost everywhere on $[0, T]$, it is reasonable that the weighted average

$$\frac{\int_0^t D(s)\mu(s)ds}{\int_0^t D(s')ds'} > 0$$

should be larger than $\mu(t) \to 0$ for sufficiently large t. We show that this does, in fact, happen if bT is sufficiently large relative to ρ in appendix K.

Example 8. *Hyperbolic case: The well-established result that a hyperbolic discount function yields a strictly concave log consumption profile follows immediately from Proposition 6.*

[27] An increasing adjusted future weight is equivalent to the property of "decreasing impatience" defined by Prelec (2004). Note that the instantaneous discount rate $\rho(t) = \rho^* - \mu(t)$ is decreasing if $\mu(t)$ is increasing. Thus decreasing impatience is a stronger condition than is necessary to get a concave log consumption profile.

From (12),

$$\varepsilon'(t) = \frac{\eta \exp(\eta t)}{1+\eta t} - \frac{\eta \exp(\eta t)}{(1+\eta t)^2} = \frac{\eta(1+\eta t)\exp(\eta t) - \eta \exp(\eta t)}{(1+\eta t)^2},$$

which is equivalent to

$$\varepsilon'(t) = \frac{\eta^2 t \exp(\eta t)}{(1+\eta t)^2}. \tag{87}$$

Thus the adjusted marginal future weighting factor for the hyperbolic discount function is

$$\begin{aligned} \mu(t) &= \frac{\varepsilon'(t)}{1+\varepsilon(t)} \\ &= \frac{\frac{\eta^2 t \exp(\eta t)}{(1+\eta t)^2}}{\frac{\exp(\eta t)}{1+\eta t}} \\ &= \frac{\eta^2 t}{1+\eta t}. \end{aligned} \tag{88}$$

Since

$$\mu'(t) = \eta^2 \frac{1+\eta t - t(\eta)}{(1+\eta t)^2} = \frac{\eta^2}{(1+\eta t)^2} > 0,$$

the adjusted marginal future weight is strictly increasing, and the log consumption profile is strictly concave. In appendix L, we show directly that both the exact and first-order concavity bounds are satisfied.

5 Terminal Consumption Revisited

In Section 3 we obtained a necessary and sufficient condition for the terminal consumption under the commitment path to be greater than the terminal consumption under the realized path. This condition took the form of a lower bound on the future weighting factor at the longest possible delay. It, in turn, is a necessary condition for the commitment path to Pareto dominate the realized path, i.e. for all selves to agree that the commitment path is at least as good as the realized path. In Section 4 we obtained a necessary and sufficient condition for the lifecycle profile of log consumption to be concave, which took the form of a lower bound on the marginal future weighting factor at each delay. Now we will show that there is a connection between what happens with the shape of the log consumption profile and the different selves' preference for the commitment path in that the condition for concavity to hold at each t is a sufficient condition for terminal consumption to be higher under the commitment path than under the realized path.

As a reminder, if $\varepsilon(t) > -1$, we can write the exact condition for concavity at $T-t$ as

$$\frac{\varepsilon'(t)}{1+\varepsilon(t)} \geq \frac{\int_0^t \exp(-\rho s)\varepsilon'(s)ds}{\int_0^t \exp(-\rho s')[1+\varepsilon(s')]ds'}.$$

Let us assume this condition holds for all $t \in [0,T]$. Then we can integrate both sides of this inequality over $[0,T]$.

$$\int_0^T \left[\frac{\int_0^t \exp(-\rho s)\varepsilon'(s)ds}{\int_0^t \exp(-\rho s')(1+\varepsilon(s'))ds'}\right] dt \leq \int_0^T \frac{\varepsilon'(t)}{1+\varepsilon(t)} dt = \ln(1+\varepsilon(T)). \tag{89}$$

Meanwhile, the exact condition (68) for $c(T|0) \geq c(T)$ can be expressed as

$$1+\varepsilon(T) \geq \exp\left(\int_0^T \left[\rho - \frac{1-\exp(-\rho t)(1+\varepsilon(t))}{\int_0^t \exp(-\rho s)(1+\varepsilon(s))ds}\right] dt\right).$$

If we rewrite that as

$$1+\varepsilon(T) \geq \exp\left(\int_0^T \left[\frac{\rho\int_0^t \exp(-\rho s')(1+\varepsilon(s'))ds' - 1 + \exp(-\rho t)(1+\varepsilon(t))}{\int_0^t \exp(-\rho s)(1+\varepsilon(s))ds}\right] dt\right),$$

we can simplify the right-hand side using (39) to obtain

$$1+\varepsilon(T) \geq \exp\left(\int_0^T \left[\frac{\rho\int_0^t \exp(-\rho s')\varepsilon(s')ds' + \exp(-\rho t)\varepsilon(t)}{\int_0^t \exp(-\rho s)(1+\varepsilon(s))ds}\right] dt\right).$$

Applying (70) to the numerator, the exact Pareto dominance condition at $t=T$ is equivalent to

$$1+\varepsilon(T) \geq \exp\left(\int_0^T \left[\frac{\int_0^t \exp(-\rho s)\varepsilon'(s)ds}{\int_0^t \exp(-\rho s')(1+\varepsilon(s'))ds'}\right] dt\right)$$

which is just (89) exponentiated. Thus if the log consumption profile is concave everywhere, terminal consumption on the commitment path must be greater than or equal to terminal consumption on the realized path. This is particularly remarkable since the commitment path played no role in our derivation of the concavity condition in Section 4.

6 Concluding remarks

In this paper we consider the problem of a finite-lived household naive about the fact that its preferences are time-inconsistent. We derive conditions on these preferences under which committing to the plan of the initial self would Pareto dominate the plan realized by the multiplicity of selves acting in sequence. In other words, all of the selves would be better off ex post if they just did what the first self wanted to do.

We know that relative discounting functions which discount utility from future consumption based on the time delay until the time that a household enjoys the consumption rather than the absolute time of the consumption (Strotz (1955)) can potentially account for several puzzles regarding the consumption and saving behavior of households. While these models are consistent with many empirical facts, the multiplicity of selves has been problematic for weighing policy proposals. Thus the conditions we derive are very useful for welfare analysis. If they hold, the household would incontrovertibly benefit from public policies that help it choose the initial consumption path. Conversely, we also obtain conditions under which the realized consumption path would Pareto dominate the initial consumption path, so policymakers would be better to leave well enough alone.

Our approach is to characterize the discount function in terms of a "future weighting factor" that measures the deviation of the discounting function from an (arbitrary) exponential discounting function. The conditions for Pareto dominance of the initial path over the realized path or vice versa largely depend on the value of the future weighting factor at the longest delay. Essentially we find that if the discounting function falls more slowly than the exponential discount function at long delays, then committing to the initial plan will be Pareto improving for all selves. If the discounting function falls much faster than the exponential at long delays, the opposite will be true.

One of the puzzles that nonexponential discounting functions can help to solve is the puzzle of why the consumption profile over the lifecycle is hump-shaped. A necessary condition for a hump-shaped consumption profile is that the log consumption profile be concave at the peak of the hump, so we also derive necessary and sufficient conditions for the log consumption profile to be concave. Perhaps our most remarkable result is the finding that the conditions for strict concavity of the log consumption profile also imply that terminal consumption is higher along the commitment path than on the realized path, which is a necessary condition for the commitment path to Pareto dominate the realized path.

In discrete time (Feigenbaum and Raei (2021)), we have shown for short life spans that the conditions for strict concavity imply that all the selves would prefer the initial plan, not just the terminal self, if the future weighting factors are small. The advantage of working in continuous time is that we can obtain exact results that do not depend on first-order approximations. The disadvantage is that welfare of all the selves except for the very last self is an integral rather than a sum, so it becomes more difficult to isolate the effect of the terminal future weighting factor. It remains an open question whether the results in discrete time can be generalized to continuous time for earlier selves.

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Appendices

A Proof of inequality (11)

We want to show for $\delta > 0$ we have

$$\frac{\delta}{1+\delta} < \ln(1+\delta),$$

or equivalently

$$\exp\left(\frac{\delta}{1+\delta}\right) < 1+\delta.$$

Let us define

$$g(\delta) = 1+\delta - \exp\left(\frac{\delta}{1+\delta}\right).$$

Then we have

$$g(0) = 0,$$

$$\begin{aligned} g'(\delta) &= 1 - \exp\left(\frac{\delta}{1+\delta}\right)\left(\frac{1}{1+\delta} - \frac{\delta}{(1+\delta)^2}\right) \\ &= 1 - \exp\left(\frac{\delta}{1+\delta}\right)\frac{1}{(1+\delta)^2}, \end{aligned}$$

and

$$\begin{aligned} g''(\delta) &= -\exp\left(\frac{\delta}{1+\delta}\right)\frac{1}{(1+\delta)^4} + 2\exp\left(\frac{\delta}{1+\delta}\right)\frac{1}{(1+\delta)^3} \\ &= \exp\left(\frac{\delta}{1+\delta}\right)\frac{2+2\delta-1}{(1+\delta)^4} \\ &= \exp\left(\frac{\delta}{1+\delta}\right)\frac{1+2\delta}{(1+\delta)^4}. \end{aligned}$$

Since $g''(\delta) > 0$ for $\delta > -\frac{1}{2}$, there can be at most one local minimum for $\delta > -\frac{1}{2}$.

$$g'(0) = 1 - \frac{1}{(1+0)^2} = 0.$$

Thus 0 is a local minimum, which is a global minimum on $\delta \geq 0$. Therefore, for $\delta > 0$, $g(\delta) > g(0) = 0$.

B Present Bias and Future Weighting Factors

Preferences exhibit a present bias if there exists an initial allocation of consumption c_0 and $c_{\Delta t}$ and consumption changes $\Delta c_0, \Delta c_{\Delta t} > 0$ such that the household would, at time 0, prefer the allocation c_0 and $c_{\Delta t}$ over $c_0 + \Delta c_0$ and $c_{\Delta t} - \Delta c_{\Delta t}$ if they are at times t and $t + \Delta t$ in the future. However, the household would prefer $c_0 + \Delta c_0$ and $c_{\Delta t} - \Delta c_{\Delta t}$ over c_0 and $c_{\Delta t}$ when the household reaches time t. In mathematical terms, this means there exists $c > 0$ such that

$$u'(c_0)\Delta c_0 - D(\Delta t)u'(c_{\Delta t})\Delta c_{\Delta t} > 0 \tag{90}$$

whereas for $t > 0$

$$D(t)u'(c_0)\Delta c_0 - D(t + \Delta t)u'(c_{\Delta t})\Delta c_{\Delta t} < 0. \tag{91}$$

We can rearrange these inequalities to get the condition

$$D(\Delta t) < \frac{u'(c_0)\Delta c_0}{u'(c_{\Delta t})\Delta c_{\Delta t}} < \frac{D(t + \Delta t)}{D(t)}. \tag{92}$$

Suppose now that (2) holds. Then (92) simplifies to

$$\varepsilon(\Delta t) < \frac{1 + \varepsilon(t + \Delta t)}{1 + \varepsilon(t)} - 1. \tag{93}$$

If the preference reversals continue in the limit as $\Delta t \to 0$,

$$\varepsilon'(0) = \lim_{\Delta t \to 0} \frac{\varepsilon(\Delta t)}{\Delta t} \leq \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[\frac{1 + \varepsilon(t + \Delta t)}{1 + \varepsilon(t)} - 1 \right] = \frac{\varepsilon'(t)}{1 + \varepsilon(t)} \tag{94}$$

For a future bias, the inequalities in (90)-(94) are all reversed.

C Simplification of $U_c(\tau)$

To show that (55) is equivalent to (56), we need to show that

$$\frac{1}{\int_t^T D(s)ds} \exp\left(- \int_0^t \frac{D(s)ds}{\int_s^T D(s'ds'} \right) = \frac{1}{\int_0^T D(s)ds}, \tag{95}$$

or equivalently that

$$\frac{\int_0^T D(s)ds}{\int_t^T D(s)ds} = \exp\left(\int_0^t \frac{D(s)ds}{\int_s^T D(s')ds'}\right).$$

If we make the substitution

$$\begin{aligned} u &= \int_s^T D(s')ds' \\ du &= -D(s)ds, \end{aligned}$$

$$\begin{aligned} \int_0^t \frac{D(s)ds}{\int_s^T D(s')ds'} &= -\int_{\int_0^T D(s)ds}^{\int_t^T D(s)ds} \frac{du}{u} \\ &= \ln\left(\int_0^T D(s)ds\right) - \ln\left(\int_t^T D(s)ds\right) \end{aligned}$$

D Derivation of $B(t,\tau)$

Let us define

$$x(t) = \ln\left(D(t)\frac{\int_0^{T-t} D(z)dz}{\int_0^T D(z')dz'}\right) + \int_0^t \frac{ds}{\int_0^{T-s} D(s')ds'}. \tag{96}$$

such that

$$\Delta U(\tau) = \int_\tau^T D(t-\tau)x(t)dt. \tag{97}$$

Using (2), we can replace $D(t)$ in (96) and rewrite it as

$$x(t) = -\rho t + \ln(1+\varepsilon(t)) + \ln\left(\int_0^{T-t} \exp(-\rho z)(1+\varepsilon(z))dz\right) \tag{98}$$

$$- \ln\left(\int_0^T \exp(-\rho z')(1+\varepsilon(z'))dz'\right) + \int_0^t \frac{ds}{\int_0^{T-s} \exp(-\rho s')(1+\varepsilon(s'))ds'}. \tag{99}$$

We can approximate $x(t)$ to first order in ε by simplifying each term in (99) as follows.

$$\ln(1+\varepsilon(t)) = \varepsilon(t) + O(\varepsilon^2).$$

$$
\begin{aligned}
\ln\left(\int_0^{T-t}\exp(-\rho z)(1+\varepsilon(z))dz\right) &= \ln\left[\int_0^{T-t}\exp(-\rho z)dz\left(1+\frac{\int_0^{T-t}\exp(-\rho z')\varepsilon(z')dz'}{\int_0^{T-t}\exp(-\rho z'')dz''}\right)\right] \\
&= \ln\left(\int_0^{T-t}\exp(-\rho z)dz\right)+\ln\left[1+\frac{\int_0^{T-t}\exp(-\rho z')\varepsilon(z')dz'}{\int_0^{T-t}\exp(-\rho z'')dz''}\right] \\
&= \ln\left(\int_0^{T-t}\exp(-\rho z)dz\right)+\frac{\int_0^{T-t}\exp(-\rho z')\varepsilon(z')dz'}{\int_0^{T-t}\exp(-\rho z'')dz''}+O(\varepsilon^2)
\end{aligned}
$$

$$
\int_0^t \frac{ds}{\int_0^{T-s}\exp(-\rho s')(1+\varepsilon(s'))ds'} = \int_0^t \frac{ds}{\int_0^{T-s}\exp(-\rho s')ds'\left(1+\frac{\int_0^{T-s}\exp(-\rho z')\varepsilon(z')dz'}{\int_0^{T-s}\exp(-\rho z)dz}\right)}
$$

$$
\begin{aligned}
\int_0^t \frac{ds}{\int_0^{T-s}\exp(-\rho s')ds'} &= \int_0^t \frac{ds}{\int_s^T \exp(-\rho^*(s'-s))ds'} \\
&= \int_0^t \frac{\exp(-\rho s)ds}{\int_s^T \exp(-\rho s')ds'}
\end{aligned}
$$

$$
\begin{aligned}
u &= \int_s^T \exp(-\rho s')ds' \\
du &= -\exp(-\rho s)ds
\end{aligned}
$$

$$
\begin{aligned}
\int_0^t \frac{ds}{\int_0^{T-s}\exp(-\rho s')ds'} &= -\int_{\int_0^T \exp(-\rho^* s)ds}^{\int_t^T \exp(-\rho s)ds}\frac{du}{u} \\
&= -\ln\left(\int_t^T \exp(-\rho s)ds\right)+\ln\left(\int_0^T \exp(-\rho s)ds\right)
\end{aligned}
$$

$$
\begin{aligned}
\ln\left(\int_0^{T-t}\exp(-\rho z)dz\right) &- \ln\left(\int_0^T \exp(-\rho z')dz'\right) \\
&= \ln\left(\int_t^T \exp(-\rho(z-t))dz\right)-\ln\left(\int_0^T \exp(-\rho z)dz\right) \\
&= \rho t+\ln\left(\int_t^T \exp(-\rho z)dz\right)-\ln\left(\int_0^T \exp(-\rho z)dz\right)
\end{aligned}
$$

Inserting these into (99), we obtain

$$x(t) = \varepsilon(t) + \frac{\int_0^{T-t} \exp(-\rho z)\varepsilon(z)dz}{\int_0^{T-t} \exp(-\rho z')dz'} - \frac{\int_0^{T} \exp(-\rho z)\varepsilon(z)dz}{\int_0^{T} \exp(-\rho z')dz'} - \int_0^t \frac{\int_0^{T-s} \exp(-\rho z)\varepsilon(z)dz}{\left(\int_0^{T-s} \exp(-\rho z')dz'\right)^2} ds + O(\varepsilon^2) \tag{100}$$

Observe that $x(t)$ vanishes when $\varepsilon(t) = 0$, so we can rewrite (97) as

$$\Delta U(\tau) = \int_\tau^T \exp(-\rho^*(t-\tau)\left[\ln\left(D(t)\frac{\int_0^{T-t} D(z)dz}{\int_0^T D(z')dz'}\right) + \int_0^t \frac{ds}{\int_0^{T-s} D(s')ds'}\right] dt + O(\varepsilon^2) \tag{101}$$

The integration set for the last term in (100) is $S = \{(s,z) : 0 \leq s \leq t \wedge 0 \leq z \leq T-s\}$. Let $S' = \{(s,z) : 0 \leq z \leq T \wedge 0 \leq s \leq \min\{t, T-z\}\}$. If $(s,z) \in S$, then $0 \leq s \leq t$ and $0 \leq z \leq T-s$. Thus $0 \leq z \leq T$. Then we have $0 \leq s$, $s \leq t$, and $s \leq T-z$. Thus $(s,z) \in S'$. If $(s,z) \in S'$, we have $0 \leq z \leq T$ and $0 \leq s \leq \min\{t, T-z\}$. Thus $0 \leq s \leq t$. We also have $0 \leq z$ and $s \leq T-z$, so $0 \leq z \leq T-s$. Thus $(s,z) \in S$.

$$\int_0^t \frac{\int_0^{T-s} \exp(-\rho z)\varepsilon(z)dz}{\left(\int_0^{T-s} \exp(-\rho z')dz'\right)^2} ds = \int_0^T \exp(-\rho z)\varepsilon(z) \int_0^{\min\{t,T-z\}} \frac{ds}{\left(\int_0^{T-s} \exp(-\rho z')dz'\right)^2} dz$$

Thus we can rewrite (97) as

$$\begin{aligned} \Delta U(\tau) &= \int_\tau^T \exp(-\rho^*(t-\tau))\Bigg[\varepsilon(t) + \frac{\int_0^{T-t} \exp(-\rho^* z)\varepsilon(z)dz}{\int_0^{T-t} \exp(-\rho^* z')dz} - \frac{\int_0^{T} \exp(-\rho^* z)\varepsilon(z)dz}{\int_0^{T} \exp(-\rho^* z')dz'} \\ &- \int_0^T \exp(-\rho^* z)\varepsilon(z) \int_0^{\min\{t,T-z\}} \frac{ds}{\left(\int_0^{T-s} \exp(-\rho^* z')dz'\right)^2} dz\Bigg] dt + O(\varepsilon^2). \end{aligned}$$

Using (62), this simplifies to

$$\Delta U(\tau) = \int_\tau^T \exp(-\rho^*(t-\tau))\left[\varepsilon(t) + \frac{\int_0^{T-t} \exp(-\rho^* z)\varepsilon(z)dz}{\int_0^{T-t} \exp(-\rho^* z')dz'} - \int_0^T \exp(-\rho^* z)\varepsilon(z)M(t,z)dz\right] dt + O(\varepsilon^2). \tag{102}$$

Then the first term in (102) is

$$\int_\tau^T \exp(\rho^*(t-\tau))\varepsilon(t)dt = \int_0^T \exp(\rho^*(z-\tau))\varepsilon(z)\Theta(z-\tau)dz.$$

The second term is

$$\begin{aligned}\int_\tau^T \exp(-\rho^*(t-\tau))\frac{\int_0^{T-t}\exp(-\rho^* z)\varepsilon(z)dz}{\int_0^{T-t}\exp(-\rho^* z')dz'} &= \int_0^T \exp(-\rho^*(t-\tau))\Theta(t-\tau)\frac{\int_0^{T-t}\exp(-\rho^* z)\varepsilon(z)dz}{\int_0^{T-t}\exp(-\rho^* z')dz'} \\ &= \int_0^T \exp(-\rho^* z)\varepsilon(z)\int_0^{T-z}\frac{\exp(-\rho^*(t-\tau))\Theta(t-\tau)}{\int_0^{T-t}\exp(-\rho^* z')dz'}dtdz\end{aligned}$$

Finally the third term is

$$\int_\tau^T \exp(-\rho^*(t-\tau))\int_0^T \exp(-\rho^* z)\varepsilon(z)M(t,z)dz = \int_0^T \exp(-\rho^* z)\varepsilon(z)\int_\tau^T \exp(-\rho^*(t-\tau))M(t,z)dtdz.$$

Thus

$$\begin{aligned}\Delta U(\tau) &= \int_0^T \exp(-\rho^* z)\varepsilon(z)\Bigg[\exp(\rho^*\tau)\Theta(z-\tau) \\ &+ \int_0^{T-z}\frac{\exp(-\rho^*(t-\tau))\Theta(t-\tau)}{\int_0^{T-t}\exp(-\rho^* z')dz'}dt - \int_\tau^T \exp(-\rho^*(t-\tau))M(t,z)dt\Bigg]dz,\end{aligned}$$

and $B(t,z)$ is the integrand divided by $\varepsilon(z)$.

E Proof of $B(z,0)=0$

$$B(z,0) = \exp(-\rho z)\left[1 + \int_0^{T-z}\frac{\exp(-\rho^* t)}{\int_0^{T-t}\exp(-\rho z')dz'}dt - \int_0^T \exp(-\rho t)M(t,z)dt\right]$$

$$\begin{aligned}\int_0^T \exp(-\rho t)M(t,z)dt &= \int_0^T \exp(-\rho^* t)\left[\frac{1}{\int_0^T \exp(-\rho z')dz'} - \int_0^{\min\{t,T-z\}}\frac{ds}{\left(\int_0^{T-s}\exp(-\rho^* z')dz'\right)^2}\right]dt \\ &= 1 - \int_0^T \int_0^{\min\{t,T-z\}}\frac{\exp(-\rho t)}{\left(\int_0^{T-s}\exp(-\rho z')dz'\right)^2}dsdt\end{aligned}$$

$$
\begin{aligned}
B(z,0) &= \exp(-\rho z)\left[\int_0^{T-z}\frac{\exp(-\rho t)}{\int_0^{T-t}\exp(-\rho z')dz'}dt - \int_0^T\int_0^{\min\{t,T-z\}}\frac{\exp(-\rho t)}{\left(\int_0^{T-s}\exp(-\rho z')dz'\right)^2}dsdt\right] \\
&= \exp(-\rho z)\left[\int_0^{T-z}\frac{\exp(-\rho t)}{\int_0^{T-t}\exp(-\rho z')dz'}dt - \int_0^T\int_0^{\min\{s,T-z\}}\frac{\exp(-\rho s)}{\left(\int_0^{T-t}\exp(-\rho z')dz'\right)^2}dtds\right]
\end{aligned}
$$

Let $S = \{(t,s) : 0 \leq s \leq T \wedge 0 \leq t \leq \min\{s, T-z\}\}$. Let $S' = \{(t,s) : 0 \leq t \leq T-z \wedge t \leq s \leq T\}$. Let $(t,s) \in S$. Then $0 \leq s \leq T$ and $0 \leq t \leq \min\{s, T-z\}$. Thus $0 \leq t \leq T-z$, and $t \leq s \leq T$, so $(t,s) \in S'$. Let $(t,s) \in S'$. Then $0 \leq t \leq T-z$ and $t \leq s \leq T$. Then $0 \leq t \leq s \leq T$, so $0 \leq s \leq T$. We also have $0 \leq t$, $t \leq T-z$, and $t \leq s$. Thus $0 \leq t \leq \min\{T-z, s\}$. Thus $(t,s) \in S$.

$$
\begin{aligned}
B(z,0) &= \exp(-\rho z)\left[\int_0^{T-z}\frac{\exp(-\rho t)}{\int_0^{T-t}\exp(-\rho z')dz'}dt - \int_0^{T-z}\int_t^{T}\frac{\exp(-\rho s)}{\left(\int_0^{T-t}\exp(-\rho z')dz'\right)^2}dsdt\right] \\
&= \exp(-\rho z)\int_0^{T-z}\frac{1}{\int_0^{T-t}\exp(-\rho z')dz'}\left[\exp(-\rho t) - \int_t^T\frac{\exp(-\rho s)ds}{\int_0^{T-t}\exp(-\rho z'')dz''}\right]dt
\end{aligned}
$$

Let $s' = s - t$. Then $ds' = ds$, and

$$
\begin{aligned}
B(z,0) &= \exp(-\rho z)\int_0^{T-z}\frac{1}{\int_0^{T-t}\exp(-\rho z')dz'}\left[\exp(-\rho t) - \frac{\int_0^{T-t}\exp(-\rho(s'+t))ds'}{\int_0^{T-t}\exp(-\rho z'')dz''}\right]dt \\
&= \exp(-\rho z)\int_0^{T-z}\frac{1}{\int_0^{T-t}\exp(-\rho z')dz'}\left[\exp(-\rho t) - \exp(-\rho t)\frac{\int_0^{T-t}\exp(-\rho s')ds'}{\int_0^{T-t}\exp(-\rho z'')dz''}\right]dt \\
&= 0.
\end{aligned}
$$

F An Alternate Expression for $c(t)$

$$
c(t) = \frac{1}{\int_0^{T-t} D(s)ds}\exp\left(-\int_0^t\frac{ds}{\int_0^{T-s} D(s')ds'}\right)R(t)W(0).
$$

In the limit as $t \to T$, the denominator vanishes. But likewise,

$$
\lim_{t\to T}\int_0^t\frac{ds}{\int_s^T D(s'-s)ds'} \to \infty,
$$

so the numerator also vanishes. Simply using l'Hôpital's rule does not help since the derivative of the numerator will evaluate to 0/0 in the limit. We can, however, rewrite the consumption function as

$$c(t) = \frac{1}{\int_0^{T-t} D(s)ds} \exp\left(-\int_0^t \frac{(1 - D(T-s))\, ds}{\int_0^{T-s} D(s')ds'} - \int_0^t \frac{D(T-s)ds}{\int_0^{T-s} D(s')ds'} \right) R(t)W(0).$$

If we make the substitution,

$$\begin{aligned} u &= \int_0^{T-s} D(s')ds' \\ du &= -D(T-s)ds, \end{aligned}$$

the second term in the exponential evaluates to

$$\begin{aligned} \int_0^t \frac{D(T-s)ds}{\int_0^{T-s} D(s')ds'} &= -\int_{\int_0^T D(s')ds'}^{\int_0^{T-t} D(s')ds'} \frac{du}{u} \\ &= -\left[\ln u\right]_{\int_0^T D(s')ds'}^{\int_0^{T-t} D(s')ds'} \\ &= \ln\left(\int_0^T D(s')ds' \right) - \ln\left(\int_0^{T-t} D(s')ds' \right) \end{aligned}$$

Thus the consumption function can also be written

$$c(t) = \frac{1}{\int_0^T D(s)ds} \exp\left(-\int_{T-t}^T \frac{(1 - D(t'))}{\int_0^{t'} D(s')ds'} dt' \right) R(t)W(0),$$

which is well-defined at $t = 0$.

G Equivalence of condition for $c(T|0) > c(T)$ in discrete and continuous time

To first order in the future weights, we showed in Section 3 that the log of terminal consumption on the commitment path will be $\varepsilon(T)$. This will also be true in discrete time as shown in Feigenbaum and Raei (2021).

Likewise, the log of terminal consumption on the equilibrium path will be

$$\ln(c(T)) = \int_0^T \left[\frac{\int_0^t \exp(-\rho s)\varepsilon'(s)ds}{\int_0^t \exp(-\rho s')ds'} \right] dt + O(\varepsilon^2).$$

Denote the first-order component of $\ln(c(T))$ by V_T. Let us make the change of variables $t' = T - t$.

$$\begin{aligned} V_T &= \int_0^T \left[\frac{\int_0^{T-t'} \exp(-\rho s)\varepsilon'(s)ds}{\int_0^{T-t'} \exp(-\rho s')ds'} \right] dt' \\ &= \int_0^T \left[\frac{\int_0^{T-t} \exp(-\rho(s+t))\varepsilon'(s)ds}{\int_0^{T-t} \exp(-\rho(s'+t))ds'} \right] dt \end{aligned}$$

Finally if we make the change of variables $z = s' + t$,

$$V_T = \int_0^T \left[\frac{\int_0^{T-t} \exp(-\rho(s+t))\varepsilon'(s)ds}{\int_t^T \exp(-\rho z)dz} \right] dt,$$

which is exactly analogous to Eq. (44) evaluated at $t = T$ in Feigenbaum and Raei (2021), the corresponding first-order expression for $\ln(c(T))$ in discrete time if we replace integrals by sums, $\exp(-\rho^*)$ by D_1, and the derivative $\varepsilon'(s)$ by the difference $\varepsilon(s+1) - \varepsilon(s)$.

H Proof of Equation (82)

$$\begin{aligned}
Z(t) &= \frac{\exp(-\rho(T-t))[1+m(T-t)]-1}{\int_0^{T-t}\exp(-\rho s)[1+ms]ds} \\
&= \frac{\exp(-\rho(T-t))[1+m(T-t)]-1}{\frac{m+\rho-\exp(-\rho(T-t))(m+(1+m(T-t))\rho)}{(\rho)^2}} \\
&= (\rho)^2\frac{\exp(-\rho(T-t))[1+m(T-t)]-1}{m+\rho-\exp(-\rho(T-t))(m+(1+m(T-t))\rho)} \\
&= (\rho)^2\frac{\exp(-\rho(T-t))[1+m(T-t)]-1}{m\,[1-\exp(-\rho(T-t))]+\rho[1-\exp(-\rho^*(T-t))(1+m(T-t))]} \\
&= -(\rho)^2\frac{1-\exp(-\rho(T-t))[1+m(T-t)]}{m\,[1-\exp(-\rho(T-t))]+\rho[1-\exp(-\rho^*(T-t))(1+m(T-t))]} \\
&= -(\rho)^2\frac{1}{\rho+m\frac{1-\exp(-\rho^*(T-t))}{1-\exp(-\rho(T-t))[1+m(T-t)]}} \\
&= -(\rho)^2\frac{1}{\rho+m\frac{1}{1-m(T-t)\frac{\exp(-\rho(T-t))}{1-\exp(-\rho(T-t))}}}
\end{aligned}$$

I Derivation of Equation (85)

$$\chi(t) = m(T-t)\left[1-\frac{1}{1-\exp(-\rho(T-t))}\right]$$

$$\begin{aligned}
\chi'(t) &= -m\left[1-\frac{1}{1-\exp(-\rho(T-t))}\right]+m(T-t)\frac{-\rho\exp(-\rho(T-t))}{(1-\exp(-\rho(T-t)))^2} \\
&= -m\left[1-\frac{1}{1-\exp(-\rho(T-t))}+\frac{\rho^*(T-t)\exp(-\rho(T-t))}{(1-\exp(-\rho(T-t)))^2}\right] \\
&= -m\frac{(1-\exp(-\rho(T-t)))^2-1+\exp(-\rho^*(T-t))+\rho(T-t)\exp(-\rho(T-t))}{(1-\exp(-\rho^*(T-t)))^2}
\end{aligned}$$

$$\chi'(t) = -m\frac{1-2\exp(-\rho(T-t))+\exp(-2\rho^*(T-t))-1+\exp(-\rho(T-t))+\rho(T-t)\exp(-\rho^*(T-t))}{(1-\exp(-\rho(T-t)))^2}$$

$$\chi'(t) = -m\frac{\exp(-2\rho(T-t))-\exp(-\rho^*(T-t))+\rho(T-t)\exp(-\rho(T-t))}{(1-\exp(-\rho^*(T-t)))^2}$$

Finally, if we factor out an exponential factor, we have

$$\chi'(t) = -m\exp(-\rho(T-t))\frac{\exp(-\rho^*(T-t)) - 1 + \rho(T-t)}{(1-\exp(-\rho(T-t)))^2}.$$

J Exact Concavity Condition with Linear Future Weights

The exact convexity condition at $T-t$ is

$$\varepsilon'(t) < \frac{\int_0^t \exp(-\rho^* s)\varepsilon'(s)ds}{\int_0^t \exp(-\rho s')[1+\varepsilon(s')]ds'}[1+\varepsilon(t)].$$

With linear future weights $\varepsilon(t) = mt$, this specializes to

$$m < \frac{m\int_0^t \exp(-\rho s)ds}{\int_0^t \exp(-\rho s')[1+ms']ds'}[1+mt].$$

Since by our definition of ρ^*, we must have $-mT < 1$,[28] we can flip the integrals to rewrite the strict concavity condition for $t > 0$ as

$$P(m,t) = \frac{m}{1+mt}\frac{\int_0^t \exp(-\rho s')[1+ms']ds'}{\int_0^t \exp(-\rho s)ds} - m < 0. \tag{103}$$

Since we can express the weighted average of $\varepsilon(s)$ over 0 to t, weighted by $\exp(-\rho^*)$, as

$$\frac{\int_0^t \exp(-\rho s')s'ds'}{\int_0^t \exp(-\rho s)ds} = \frac{1}{\rho}\left[1 - \frac{\rho t}{\exp(\rho^* t)-1}\right], \tag{104}$$

this simplifies to

$$P(m,t) = \frac{m}{1+mt}\left[1+\frac{m}{\rho}\left(1-\frac{\rho t}{\exp(\rho t)-1}\right)\right] - m. \tag{105}$$

[28] In what follows, it does not actually matter that we represent $D(t) = \exp(-\rho^* t)(1+\varepsilon(t))$. Any representation $\exp(-\rho t)(1+\varepsilon(t))$ would work the same as long as $\varepsilon(t) \geq 0$ for all t.

Obviously, $P(0,t) = 0$ for $t > 0$. The partial derivative of $P(m,t)$ with respect to m is

$$\begin{aligned}\frac{\partial P}{\partial m}(m,t) &= \frac{m}{1+mt}\frac{1}{\rho}\left(1-\frac{\rho t}{\exp(\rho t)-1}\right)+\left(\frac{(1+mt)-mt}{(1+mt)^2}\right)\left[1+\frac{m}{\rho}\left(1-\frac{\rho t}{\exp(\rho t)-1}\right)\right]-1\\ &= \frac{m}{1+mt}\frac{1}{\rho}\left(1-\frac{\rho t}{\exp(\rho t)-1}\right)+\frac{1}{(1+mt)^2}\left[1+\frac{m}{\rho}\left(1-\frac{\rho t}{\exp(\rho t)-1}\right)\right]-1.\end{aligned}$$

Thus $\frac{\partial P}{\partial m}(0,t)$ also vanishes for $t > 0$.

Combining terms we obtain

$$\frac{\partial P}{\partial m}(m,t) = \left[\frac{m}{1+mt}+\frac{m}{(1+mt)^2}\right]\frac{1}{\rho^*}\left(1-\frac{\rho^* t}{\exp(\rho^* t)-1}\right)+\frac{1}{(1+mt)^2}-1.$$

Note that

$$\begin{aligned}\frac{\partial}{\partial m}\left(\frac{m}{1+mt}+\frac{m}{(1+mt)^2}\right) &= \frac{(1+mt)-mt}{(1+mt)^2}+\frac{1}{(1+mt)^2}-\frac{2mt}{(1+mt)^3}\\ &= \frac{2}{(1+mt)^2}-\frac{2mt}{(1+mt)^3}\\ &= \frac{2}{(1+mt)^3}.\end{aligned}$$

Plugging this in, we get the second partial derivative

$$\begin{aligned}\frac{\partial^2 P}{\partial m^2}P(m,t) &= \frac{2}{(1+mt)^3}\frac{1}{\rho^*}\left(1-\frac{\rho^* t}{\exp(\rho^* t)-1}\right)-\frac{2t}{(1+mt)^3}\\ &= \frac{2}{(1+mt)^3}\frac{1}{\rho}\left(1-\rho t\frac{1+\exp(\rho t)-1}{\exp(\rho t)-1}\right)\\ &= \frac{2}{(1+mt)^3}\frac{1}{\rho}\left(1-\frac{\rho^* t}{1-\exp(-\rho^* t)}\right).\end{aligned}$$

Using (7), we have for $t > 0$

$$\frac{\partial^2 P}{\partial m^2}(m,t) = \frac{2}{(1+mt)^3}\frac{1}{\rho}\frac{1-\rho t-\exp(-\rho^* t)}{1-\exp(-\rho t)} < 0. \tag{106}$$

Thus $P(m) < 0$ for $m \neq 0$, and the strict convexity condition holds for $t > 0$ if ε is linear.

K The Shape of the Log Consumption Profile with Quadratic Future Weights

Let us consider a quadratic form for $\varepsilon(t)$

$$\varepsilon(t) = mt + \frac{1}{2}bt^2,$$

so we have

$$\varepsilon'(t) = m + bt.$$

Then a sufficient condition to first-order in ε for strict concavity of the log consumption profile can be written as

$$\varepsilon'(t) > \frac{\int_0^t \exp(-\rho z)(m+bz)dz}{\int_0^t \exp(-\rho z')dz'}.$$

Since

$$\int_0^t \exp(-\rho z)(m+bz)dz = \frac{b + m\rho - \exp(-\rho t)(b + (m+bt)\rho)}{(\rho)^2}, \tag{107}$$

we can, using (39), write the concavity bound as

$$\begin{aligned} m + bt &> \frac{\frac{b+m\rho-\exp(-\rho^* t)(b+(m+bt)\rho)}{(\rho)^2}}{\frac{1-\exp(-\rho^* t)}{\rho}} \\ &= m + \frac{b}{\rho}\frac{1-(1+\rho t)\exp(-\rho t)}{1-\exp(-\rho t)}. \end{aligned} \tag{108}$$

Cancelling the ms, the inequality is trivially false if $b = 0$. If $b > 0$, then we can also cancel the bs, and we have

$$\rho^* t(1 - \exp(-\rho^* t)) > 1 - (1 + \rho^* t)\exp(-\rho^* t).$$

This simplifies to

$$\exp(-\rho^* t) > 1 - \rho^* t,$$

which is true by (7) for all $t > 0$. Note that if $b < 0$ then the inequality (108) will be reversed when we cancel the bs. In that case, the first-order condition for strict convexity will be satisfied for all $t > 0$.

Let us now consider the exact concavity condition. As we show in the text, if both m

and b are nonzero, the exact concavity condition will not hold for t sufficiently small that $\varepsilon(t)$ is effectively linear, so we only consider the case where $m = 0$ and $b \neq 0$.

Then the adjusted marginal future weight will be

$$\mu(t) = \frac{\varepsilon'(t)}{1+\varepsilon(t)} = \frac{bt}{1+\frac{1}{2}bt^2}. \tag{109}$$

The right-hand side of the concavity bound (86) is

$$B(t) = \frac{\int_0^t \exp(-\rho^* z) bz dz}{\int_0^t \exp(-\rho^* z')(1+\frac{1}{2}b(z')^2)dz'}. \tag{110}$$

From (107), the numerator is

$$b\frac{1-\exp(-\rho^* t)(1+\rho^* t)}{(\rho^*)^2}.$$

The denominator is

$$\frac{1-\exp(-\rho^* t)}{\rho^*} + b\frac{1-\exp(-\rho^* t)(1+\rho^* t+\frac{1}{2}(\rho^* t)^2)}{(\rho^*)^3}.$$

The log consumption profile will be concave at $T-t$ iff $\mu(t) \geq B(t)$, or equivalently if

$$\frac{t}{1+\frac{1}{2}bt^2} \geq \frac{\rho^*\left[1-\exp(-\rho^* t)(1+\rho^* t)\right]}{(\rho^*)^2\left[1-\exp(-\rho^* t)\right] + b\left[1-\exp(-\rho^* t)(1+\rho^* t+\frac{1}{2}(\rho^* t)^2)\right]}$$

. That is if

$$\begin{aligned}(\rho^*)^2 t\left[1-\exp(-\rho^* t)\right] &+ bt\left[1-\exp(-\rho^* t)\left(1+\rho^* t+\frac{1}{2}(\rho^* t)^2\right)\right] \\ &\geq \rho^*\left[1-\exp(-\rho^* t)(1+\rho^* t)\right]\left(1+\frac{1}{2}bt^2\right).\end{aligned}$$

If we put the exponential terms on the right-hand side and everything else on the left, we obtain

$$\begin{aligned}
(\rho^*)^2 t + bt &- \rho^* - \frac{1}{2} b\rho^* t^2 \\
&\geq \exp(-\rho^* t)\left[(\rho^*)^2 t + bt\left(1 + \rho^* t + \frac{1}{2}(\rho^* t)^2\right) - \rho^*(1+\rho^* t)\left(1 + \frac{1}{2}bt^2\right)\right] \\
&= \exp(-\rho^* t)\left[bt + \frac{1}{2} b\rho^* t^2 - \rho^*\right].
\end{aligned}$$

Thus

$$\rho^* - bt \geq \frac{1-\exp(-\rho^* t)}{\rho^* t}\left[\rho^* - bt - \frac{1}{2} b\rho^* t^2\right].$$

Isolating $\rho^* t - b$, this becomes

$$\left(\frac{\rho^* t}{1-\exp(-\rho^* t)} - 1\right)(\rho^* - bt) \geq -\frac{1}{2} b\rho^* t^2.$$

In other words, the log consumption profile will be concave at $T - t$ iff

$$\frac{1}{2} b\rho^* t^2 \geq \frac{1 - \rho^* t - \exp(-\rho^* t)}{1-\exp(-\rho^* t)}(\rho^* - bt).$$

Suppose $b > 0$. Then the left-hand side will be strictly positive for $t > 0$. By (7), the fraction on the right-hand side will be strictly negative for $t > 0$. Thus for small t, the inequality will be satisfied strictly, and the log consumption profile will be strictly concave. However if $bt > \rho^a st$, the right-hand side will become positive. In the limit of large t, the right-hand side will go as $b\rho^* t^2$, so the inequality will be violated, and the log consumpion profile will become convex.

L The Shape of the Log Consumption Profile with Hyperbolic Discounting

Concavity bound to first order in ε: To first order in ε, the concavity condition is

$$\varepsilon'(t) > B_1(t) = \frac{\int_0^t \exp(-\rho^* z)\varepsilon'(z)dz}{\int_0^t \exp(-\rho z')dz'}.$$

From (87), the numerator is

$$\begin{aligned}\int_0^t \exp(-\rho s)\varepsilon'(s)ds &= \int_0^t \frac{\eta^2 s}{(1+\eta s)^2}ds \\ &= \int_0^t \left[\frac{\eta}{1+\eta s} - \frac{\eta}{(1+\eta s)^2}\right] ds \\ &= \int_1^{1+\eta t} \left[\frac{1}{u} - \frac{1}{u^2}\right] du \\ &= \ln(1+\eta t) + \frac{1}{1+\eta t} - 1. \qquad (111)\end{aligned}$$

Using (39) to simplify the denominator, the first-order concavity bound is

$$\begin{aligned}B_1(t) &= \frac{\ln(1+\eta t) + \frac{1}{1+\eta t} - 1}{\frac{1-\exp(-\eta t)}{\eta}} \\ &= \eta \frac{\ln(1+\eta t) + \frac{1}{1+\eta t} - 1}{\exp(\eta t) - 1} \exp(\eta t). \qquad (112)\end{aligned}$$

Thus, applying (87) again, the log consumption profile will be concave to first order iff

$$\frac{\eta^2 t \exp(\eta t)}{(1+\eta t)^2} \geq \eta \frac{\ln(1+\eta t) + \frac{1}{1+\eta t} - 1}{\exp(\eta t) - 1} \exp(\eta t). \qquad (113)$$

If we define $x = \eta t$ with equality, so the condition is

$$\frac{x}{(1+x)^2} \geq \frac{\ln(1+x) - \frac{x}{1+x}}{\exp(x) - 1}.$$

For $x > 0$, this can be rearranged to

$$x[\exp(x) + x] \geq (1+x)^2 \ln(1+x). \qquad (114)$$

But since, for $x > 0$,

$$\exp(x) > 1 + x + \frac{x^2}{2},$$

we have

$$x(\exp(x) + x) > x + 2x^2 + \frac{x^3}{2}.$$

Meanwhile, by Taylor's theorem,[29]

$$(1+x)^2 \ln(1+x) < x + \frac{3}{2}x^2 + \frac{x^3}{3}.$$

Thus (113) holds strictly for $x > 0$, and the log consumption profile is strictly concave to first order.

Surprisingly, the corresponding comparison for the exact concavity bound

$$B(t) = \frac{\int_0^t \exp(-\rho^* z)\varepsilon'(z)dz(1+\varepsilon(t))}{\int_0^t \exp(-\rho z')(1+\varepsilon(z'))dz'}$$

is more straightforward than the preceding comparision for the approximate bound. From (12), the denominator is now

$$\begin{aligned}\int_0^t \exp(-\eta z)(1+\varepsilon(z))dz &= \int_0^t \frac{dz}{1+\eta z} \\ &= \frac{\ln(1+\eta t)}{\eta}.\end{aligned}$$

Thus

$$B(t) = \eta \frac{\ln(1+\eta t) + \frac{1}{1+\eta t} - 1}{\ln(1+\eta t)} \frac{\exp(\eta t)}{1+\eta t}. \tag{115}$$

Using (87), we can rewrite this as

$$\begin{aligned}B(t) &= \eta \frac{\ln(1+\eta t) + \frac{1}{1+\eta t} - 1}{\ln(1+\eta t)} \frac{1+\eta t}{\eta t} \\ &= \frac{\frac{1+\eta t}{\eta t}\ln(1+\eta t) + \frac{1}{\eta t} - \frac{1+\eta t}{\eta t}}{\ln(1+\eta t)} \varepsilon'(t) \\ &= \left[\frac{1}{\eta t} + 1 - \frac{1}{\ln(1+\eta t)}\right] \varepsilon'(t). \end{aligned} \tag{116}$$

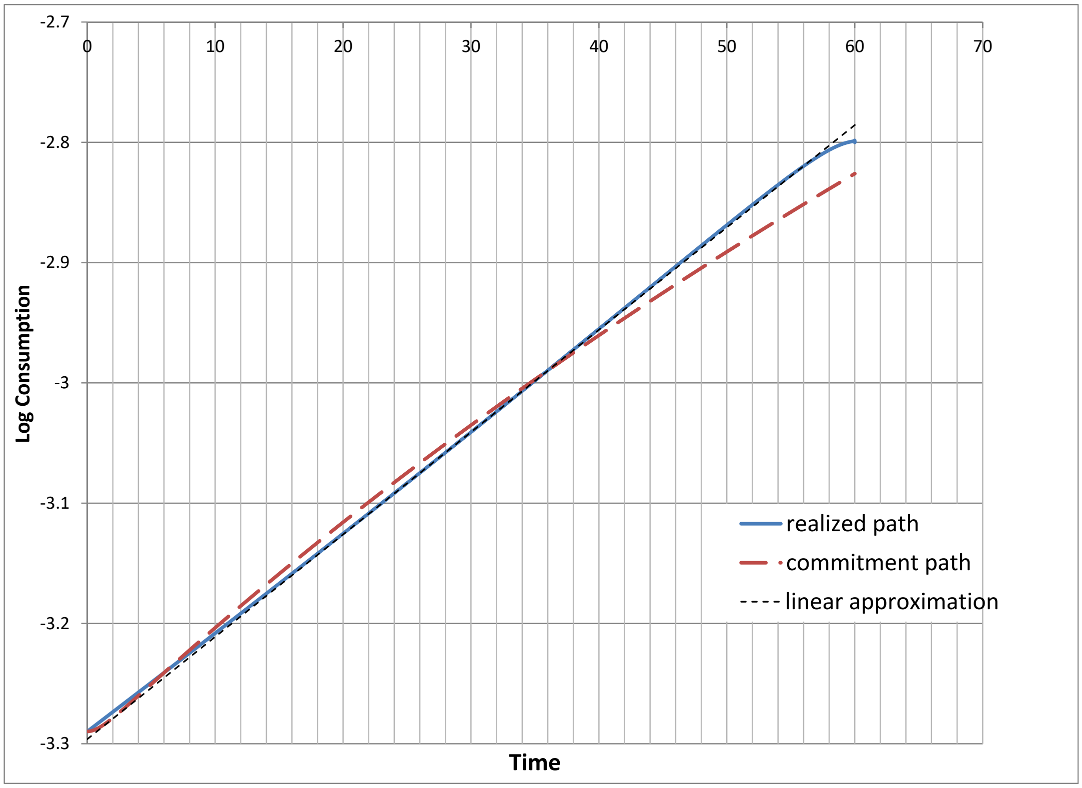
Taking the log of (7), we have

$$\ln(1+\eta t) \le \eta t$$

with equality only if $t = 0$. Thus the term in brackets in (116) is strictly less than 1 for $t > 0$, so the exact concavity bound is strictly satisfied with hyperbolic discounting.

[29]If $g_h(x) = (1+x)^2 \ln(1+x)$, $g_h(0) = 0$, $g_h'(0) = 1$, $g_h''(0) = 3$, $g_h'''(0) = 2$, and $g_h^{(4)} < 0$.

Figure 1: Comparison of log consumption for commitment path and realized path



Note: This figure compares the lifecycle profiles of log consumption for the realized and commitment paths in the particular case where $T = 60$, $\lambda = 1.0$, $\eta = 0.01$, and $\rho = r = 0.04$..