A Nonparametric Formula Relating the Elasticity of a Factor Demand to the Elasticity of Substitution*

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January 4, 2019

Abstract

It is well known for a Cobb-Douglas production function that the elasticity of a factor demand is the inverse of the share of output going to the other factors. Since Cobb-Douglas has a unit elasticity of substitution, the demand elasticity trivially equals the ratio of the elasticity of substitution to the share of output going to the other factor. I show here that this result can be generalized to any constant returns to scale production function.

JEL Classification: J23

Keywords: elasticity of substitution, elasticity of labor demand, output share

Between 1984 and 2017, real median household income in the United States increased by 0.6% per year.[1] During the same period, real per capita GDP increased by 1.6% per year.[2] Much has been written about the deficit between these two income growth rates. One possible culprit for the stagnation of most workers' wages that gets considerable attention, albeit mostly from laypeople and politicians, is excessive immigration, especially illegal immigration. Of course, economists understand that the degree to which an influx of workers will depress wages depends on the elasticity of demand for labor, but estimating supply and demand functions is a difficult econometric exercise, which muddies the policy debate. Here I derive a simple nonparametric formula that expresses

*I would like to thank Scott Findley, Martin Gervais, Pedro Silos, and Josh Smith for their comments and advice.

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[1] https://fred.stlouisfed.org/series/MEHOINUSA672N

[2] https://fred.stlouisfed.org/series/GDPCA and https://fred.stlouisfed.org/series/B230RC0A052NBEA

the elasticity of demand for a factor in terms of variables that are observable or, at least, boundable.

The elasticity of demand will equal the elasticity of substitution between the factor and any other factors divided by the share of output going to those other factors. To derive this we need only assume that the production function exhibits constant returns to scale and factor markets are perfectly competitive. It is not surprising that the elasticity of demand for a good should depend on the elasticity of substitution between the good and its alternatives, though it is remarkable that the two elasticities are exactly proportional. However, the formula is also consistent with the common textbook intuition that the demand for a good becomes less elastic as you define it more narrowly. It is already well known that the demand for a factor will become perfectly elastic if the factor is the sole input in production since the marginal product must be constant if there is constant returns to scale. What is not so obvious is that the elasticity of demand decreases monotonically with the share of output going to the factor, converging to the elasticity of substitution in the limit where the factor's contribution to output is vanishingly small.

Like the elasticity of demand, the elasticity of substitution can be difficult to measure. But if there is high confidence that two factors are substitutes (or complements), we can bound the elasticity of their demands. Returning to the example of wages for low-skilled workers, most evidence suggests that low-skilled labor is a substitute for more technologically sophisticated factors.[3] Assuming that the relevant elasticity of substitution is greater than 1, this will be a lower bound on the elasticity of demand for low-skilled labor. Thus a 1% increase in the supply of low-skilled labor can, at most, cause a 1% decrease in low-skilled wages. There are simply not enough immigrants in America to explain a 40% deficiency in median wages, compounded over three decades.

The paper proceeds as follows. In Section 1, we review the familiar example of a Cobb-Douglas production function and demonstrate that the formula is trivially satisfied in this special case. In Section 2, we derive the formula for a general production function with two factors and constant returns to scale. In Section 3, we show how a production function with more than two factors can effectively be reduced to one with two factors, preserving the utility of the formula. We conclude in Section 4 with a discussion of how the formula applies to labor markets.

1 Cobb-Douglas Production

Suppose we just have two inputs Z_1 and Z_2 with factor prices p_1 and p_2, and the production function is $H(Z_1, Z_2)$. If factor markets are competitive, in equilibrium each factor price must equal the corresponding marginal product:

$$p_i = H_i(Z_1, Z_2). \tag{1}$$

[3] See, for example, Acemoglu and Restrepo (2017).

To begin with, let us review the familiar case of a Cobb-Douglas production function

$$H(Z_1, Z_2) = Z_1^{\alpha} Z_2^{1-\alpha}, \tag{2}$$

where $\alpha \in [0, 1]$. The marginal products will be

$$p_1 = H_1(Z_1, Z_2) = \alpha \left(\frac{Z_1}{Z_2} \right)^{\alpha - 1} \tag{3}$$

and

$$p_2 = H_2(Z_1, Z_2) = (1 - \alpha) \left(\frac{Z_1}{Z_2} \right)^{\alpha}. \tag{4}$$

Thus the ratio of the factor prices will be

$$\frac{p_1}{p_2} = \frac{\alpha}{1 - \alpha} \frac{Z_2}{Z_1}. \tag{5}$$

Thus the elasticity of substitution is

$$\xi = \frac{\partial \ln (Z_1/Z_2)}{\partial \ln(p_2/p_1)} = 1. \tag{6}$$

Note also that (1) implies that the share of output going to the first factor is

$$\frac{p_1 Z_1}{H(Z_1, Z_2)} = \frac{\alpha Z_1^{\alpha} Z_2^{1-\alpha}}{Z_1^{\alpha} Z_2^{1-\alpha}} = \alpha. \tag{7}$$

From (4), we see that the elasticity of demand for the second factor is

$$\frac{\partial \ln(Z_2)}{\partial \ln(p_2)} = -\frac{1}{\alpha} = -\left(\frac{p_1 Z_1}{H(Z_1, Z_2)} \right)^{-1}. \tag{8}$$

2 Elasticity of Factor Demand for a General Production Function with Two Factors

Now let us generalize to the case of any constant returns to scale production function H. Because H has constant returns to scale, we can rewrite it as

$$H(Z_1, Z_2) = Z_2 h(z), \tag{9}$$

where

$$h(z) = H(z, 1) \tag{10}$$

is the intrinsic production function and

$$z = \frac{Z_1}{Z_2} \tag{11}$$

is the ratio of the two factors. Then the marginal product of Z_1 reduces to just the derivative of h:

$$\frac{\partial}{\partial Z_1}\left(Z_2 h\left(\frac{Z_1}{Z_2}\right)\right) = Z_2 h'\left(\frac{Z_1}{Z_2}\right)\frac{1}{Z_2} = h'(z). \tag{12}$$

Likewise, the marginal product of Z_2 is

$$\frac{\partial}{\partial Z_2}\left(Z_2 h\left(\frac{Z_1}{Z_2}\right)\right) = h\left(\frac{Z_1}{Z_2}\right) + Z_2 h'\left(\frac{Z_1}{Z_2}\right)\left(-\frac{Z_1}{Z_2^2}\right) = h(z) - zh'(z). \tag{13}$$

Thus we can express the factor prices as functions of z:

$$p_1 = h'(z) \tag{14}$$
$$p_2 = h(z) - zh'(z), \tag{15}$$

and the ratio of the factor prices is

$$\frac{p_2}{p_1} = \frac{h(z)}{h'(z)} - z.$$

$$\frac{\partial\left(\frac{p_2}{p_1}\right)}{\partial z} = \frac{h'(z)}{h'(z)} - \frac{h(z)h''(z)}{(h'(z))^2} - 1 = -\frac{h(z)h''(z)}{(h'(z))^2}$$

Let $\xi(z)$ denote the elasticity of substitution between the two factors. Then

$$\xi(z)^{-1} = \frac{\partial \ln\left(\frac{p_2}{p_1}\right)}{\partial \ln z} = -\frac{h(z)h''(z)}{(h'(z))^2}\frac{z}{\frac{h(z)}{h'(z)} - z} = -\frac{zh''(z)}{h'(z)}\frac{h(z)}{h(z) - h'(z)z}. \tag{16}$$

Let us denote the share of output going to Z_1 by

$$\alpha(z) = \frac{p_1 Z_1}{H(Z_1, Z_2)} = \frac{h'(z)Z_1}{Z_2 h(z)} = \frac{zh'(z)}{h(z)}. \tag{17}$$

Our question is how factor prices vary with the supply of the corresponding factor. The response of p_2 to a change in Z_2 is

$$\frac{\partial p_2}{\partial Z_2} = \frac{\partial}{\partial Z_2}\left(h\left(\frac{Z_1}{Z_2}\right) - \frac{Z_1}{Z_2}h'\left(\frac{Z_1}{Z_2}\right)\right) \tag{18}$$

This simplifies to

$$\frac{\partial p_2}{\partial Z_2} = h'(z)\left(-\frac{Z_1}{Z_2^2}\right) + \frac{Z_1}{Z_2^2}h'(z) - zh''(z)\left(-\frac{Z_1}{Z_2^2}\right) = \frac{z^2}{Z_2}h''(z)$$

Thus the elasticity of p_2 over Z_2 is

$$\begin{aligned}\frac{\partial \ln(p_2)}{\partial \ln(Z_2)} &= \frac{Z_2}{p_2}\frac{\partial p_2}{\partial Z_2} = \frac{Z_2}{h(z) - zh'(z)}\left(\frac{z^2}{Z_2}h''(z)\right) = \frac{z^2 h''(z)}{h(z) - zh'(z)} \\ &= \frac{zh''(z)}{h(z) - zh'(z)}z \\ &= -\left(-\frac{zh''(z)}{h'(z)}\frac{h(z)}{h(z) - h'(z)z}\right)\frac{zh'(z)}{h(z)}\end{aligned}$$

$$\frac{\partial \ln(p_2)}{\partial \ln(Z_2)} = -\frac{\alpha(z)}{\xi(z)}. \tag{19}$$

Thus the elasticity of the price of factor 2 with respect to the supply of factor 2 is the output share of factor 1 divided by the elasticity of substitution between the two factors. Alternatively, the elasticity of demand for Z_2 is

$$\frac{\partial \ln(Z_2)}{\partial \ln(p_2)} = -\frac{\xi(z)}{\alpha(z)}. \tag{20}$$

3 Generalizing to Three or More Factors

Now suppose that we have a constant returns to scale production function F of $n+1$ factors, where $n \geq 2$. Let us denote these factors $X_1, \ldots, X_n$, and Z_2 and the corresponding factor prices by $q_1, \ldots, q_n$, and p_2. We can still make use of the result from Section 2 by constructing a composite factor equal to the total expenditure on the inputs X_1 and X_2:

$$Z_1 = \sum_{i=1}^{n} q_i X_i. \tag{21}$$

Let us define the effective production function[4]

$$H(Z_1, Z_2) = \max_{X_1,\ldots,X_n} F(X_1, \ldots, X_n, Z_2) \tag{22}$$

subject to

$$\sum_{i=1}^{n} q_i X_i = Z_1. \tag{23}$$

Proposition 1 *H will exhibit constant returns to scale.*

[4] For the common special case in which $F(X_1, ..., X_n, Z_2) = \widetilde{H}(G(X_1, ..., X_n), Z_2)$, where both $\widetilde{H}$ and G exhibit constant returns to scale, we will have $H = \widetilde{H}$ and $Z_1 = G(X_1, ..., X_n)$.

Let $\lambda > 0$. Then

$$H(\lambda Z_1, \lambda Z_2) = \max_{X_1,\dots,X_n} F(X_1, \dots, X_n, \lambda Z_2)$$

subject to

$$\sum_{i=1}^{n} q_i X_i = \lambda Z_1.$$

Thus

$$H(\lambda Z_1, \lambda Z_2) = \max_{X_1,\dots,X_n} \lambda F\left(\frac{X_1}{\lambda}, \dots, \frac{X_n}{\lambda}, Z_2\right)$$

subject to

$$\sum_{i=1}^{n} q_i \frac{X_i}{\lambda} = Z_1.$$

Let us define $x_i = \frac{X_i}{\lambda}$. Then

$$H(\lambda Z_1, \lambda Z_2) = \max_{x_1,\dots,x_n} \lambda F\left(x_1, \dots, x_n, Z_2\right)$$

subject to

$$\sum_{i=1}^{n} q_i x_i = Z_1$$

But the x_i and X_i are just dummy variables, so

$$H(\lambda Z_1, \lambda Z_2) = \lambda H(Z_1, Z_2). \tag{24}$$

Since the price of Z_1 is by construction 1, the elasticity of demand for Z_2 will be, from Section 2,

$$\frac{\partial \ln Z_2}{\partial \ln p_2} = \frac{H(Z_1, Z_2)}{Z_1} \xi(Z_1, Z_2), \tag{25}$$

where $\xi(Z_1, Z_2)$ is the elasticity of substitution between Z_2 and the composite factor Z_1

4 An Application to Labor Markets

As an example of how this formula can be useful, consider a macroeconomic model in which capital and labor are the factors of production, but labor can be differentiated into multiple types.[5] If factor markets are competitive, the elasticity of demand for each type of labor will depend on both the share of output going to that labor type and the elasticity of substitution between this labor type and the composite of all other factors.

[5] We could also differentiate capital.

The simplest nontrivial case has two skill levels of labor. For example, Krussel et al (2000) employ a specification exhibiting capital-skill complementarity, where capital and high-skilled labor have a constant elasticity of substitution less than 1. Meanwhile, the combination of high-skilled labor and capital has a constant elasticity of substitution with low-skilled labor that is greater than 1. The nonparametric formula (25) allows us to generalize this result to any constant returns to scale production function. As long as low-skilled labor is a substitute for high-skilled labor and/or capital, so the elasticity of substitution is greater than one, we can infer that the elasticity of demand for low-skilled labor will be bounded from below (in absolute value) by the inverse of the output share of high-skilled labor and capital. This in turn is bounded from below by one. Conversely, the wage elasticity of demand for low-skilled labor will be bounded from above by one. Thus a 1% increase in the supply of low-skilled labor can at most cause a 1% decrease in low-skilled wages.

This is roughly consistent with Borjas' (2017) estimate of the wage elasticity of demand for very low-skilled labor in Miami during the Mariel boat lift. Looking specifically at high school dropouts, he found the wage elasticity to lie between 0.5 and 1.5.[6] For a very narrow definition of low-skilled labor, the share of output going to other factors will essentially be 1. Thus the wage elasticity will just be the inverse of the elasticity of substitution between high school dropouts and other inputs. That elasticity of substitution could be less than 1 for essential services of manual labor that presently have no convenient technological substitutes, such as janitorial services or truck driving. The boat lift also occurred in the 70s and 80s when fewer jobs had been replaced by technology.

References

[1] Acemoglu, Daron and Pascual Restrepo, (2017), "Robots and Jobs: Evidence from US Labor Markets," NBER Working Paper No. 23285.

[2] Borjas, George J., (2017), "The Wage Impact of the *Marielitos*: A Reappraisal," *Industrial and Labor Relations Review* 70: 1077-1110.

[3] Clemens, Michael A. and Jennifer Hunt, (2017), "The Labor Market Effects of Refugee Waves: Reconciling Conflicting Results," NBER Working Paper No. 23433.

[4] Krusell, Per, Lee E. Ohanian, José-Víctor Ríos-Rull, and Giovanni L. Violante, (2000), "Capital-Skill Complementarity and Inequality: A Macroeconomic Analysis," *Econometrica* 68: 1029-1053.

[6] Clemens and Hunt (2017) argue that Borjas' estimates are biased upward, which would bring the wage elasticity even more into line with the nonparametric formula.