How the Future Shapes Consumption with Time-Inconsistent Preferences*

James Feigenbaum† Sepideh Raei‡

January 12, 2022

Abstract

Time-inconsistent preferences, which are modeled by relative discount functions, are a common explanation for the empirical finding that lifecycle profiles of household consumption are typically hump-shaped rather than monotonic. Such time-inconsistency is often described in terms of a present or future bias, which characterizes how preferences regarding trade-offs between consumption in the near future vs the far future get revised as the household approaches the final decision point for this trade-off. Only an exponential discount function is immune to preference revisions. Using finite-lived, naive households with log utility, we assess how the practical consequences of time-inconsistent preferences depend on the deviation of the discount function from an exponential function, which we measure in terms of a perturbing parameter called the *future weighting factor*. We derive necessary and sufficient conditions on the future weighting factors for the consumption profile to be locally concave. These conditions, which are necessary for the consumption profile to be hump-shaped, are stronger than just assuming a present bias. We also obtain necessary and sufficient conditions under which the consumption profile determined in the first period of life (almost) Pareto dominates the realized consumption profiles. For short lifespans, we can show the conditions for concavity of the whole log consumption profile imply almost Pareto dominance.

JEL: D60, D90

Keywords: present bias, future bias, time-inconsistent preferences, consumption hump, commitment mechanisms

---

*We would like to thank Frank Caliendo and Scott Findley for their input.

†Utah State University, John Huntsman School of Business, Utah, United States; james.feigenbaum@usu.edu.

‡Utah State University, John Huntsman School of Business, Utah, United States; sepideh.raei@usu.edu.

1 Introduction

The canonical life-cycle model predicts that consumption will grow smoothly for patient individuals and decay smoothly for impatient individuals. However, from an empirical standpoint, one of the most striking aspects of people's choices of consumption over the lifecycle is that this profile is generally hump-shaped. As was first documented by Thurow (1969), average consumption increases while consumers are young, peaks when they reach middle age, and decreases afterwards.[1] In this paper, we propose a general representation of the discount function that nests all discount functions. Using this representation, we establish conditions under which the consumption profile is consistent with the empirical evidence. In a nutshell, whether a hump-shaped consumption profile can occur will depend on precisely how the discount function deviates from an exponential discount function.

A sizeable literature is devoted to developing theoretical frameworks that modify the Lifecycle/Permanent-Income Hypothesis of Friedman and Modigliani (Modigliani and Brumberg (1954), Friedman (2018)) to address the inconsistency between model predictions for consumption profile and empirical evidence, which is often referred to as the "lifecycle consumption puzzle".[2] A more traditional set of solutions to this puzzle would include family-size effects (Attanasio et al. (1999), Attanasio and Browning (1993), Browning et al. (1985)), consumption-leisure trade-offs (Heckman (1974), Bullard and Feigenbaum (2007)), wage income uncertainty and the precautionary saving motive (Nagatani (1972), Hubbard et al. (1994), Carroll (1994), Carroll (1997), Gourinchas and Parker (2002)), mortality risk (Feigenbaum (2008), Hansen and Imrohoroglu (2008)), and consumer durables (Fernandez-Villaverde and Krueger (2011)).

Another set of solutions which provide somewhat more robust explanations for the hump in the consumption profile relax the assumptions of the standard rational paradigm, epitomized by Samuelson (1937).[3] One of the most popular of these is to allow for time-inconsistent preferences by generalizing the discount function from an exponential function. Strotz (1955b) was the first to explore such deviations from Samuelson's model. Phelps and Pollak (1968) later proposed the hyperbolic function as a specific alternative to the exponential function, and David Laibson's dissertation (Laibson (1994)) offered hyperbolic

[1]See Carroll and Summers (1991), Attanasio and Weber (1995), Attanasio et al. (1999), Browning and Crossley (2001), Gourinchas and Parker (2002), and Fernandez-Villaverde and Krueger (2011).

[2]See Deaton (1992) and Browning and Crossley (2001) for more recent overviews.

[3]The preceding explanations often yield wildly different predictions in response to small changes in the economic environment, such as the introduction of Social Security.

discounting as a solution to the consumption hump puzzle. Today, this strand of the literature generally attributes such consumption humps to the concept of "present bias".[4]

Present bias, or as Ericson and Laibson (2019) called it present focus,[5] is a form of time-inconsistency in which individuals are more impatient in trade-offs between the present and the immediate future as compared to trade-offs between equivalent intervals of time in the more distant future. Individuals acting under this bias, who might have been inclined to postpone a future payoff when the options for when to take it were all in the future, are more inclined to take it at the first opportunity as the opportunity gets closer tp present.[6] Such a change is referred to as a preference reversal. Note that with exponential discounting, the concept of time can be assumed to be either absolute time, calendar time, or even waiting time, i.e. the time to consumption. As Strotz (1955a) showed, the equivalence of these three temporal interpretations is a consequence of the exponential function not exhibiting preference reversals. In contrast, for a nonexponential discount function that exhibits present bias, we must interpret the "time" that parameterizes the function as the delay or waiting time until we experience the consumption from the present moment.

In this paper, we propose a general representation of a discount function in the form of $D_t = D_1^t(1+\varepsilon_t)$ for $t = 1, \ldots, T$, where ε_t is the **extra weight** (compared to the exponential discounting case) that we put on the discount factor t periods in the future, and $T + 1$ is the life span. We call ε_t the *future weighting factor*. All forms of discount function, including the nonexponential ones, can be written as a specific case of this general function by finding the corresponding ε_t. An advantage of this novel approach is the opportunity it provides to understand the driving force behind the consumption hump. Surprisingly, given the emphasis of the literature on present bias, using this approach we find that the shape of the consumption profile at a given age depends on the dynamics of the future weighting factors at all delays within the remaining time horizon of the household, rather than just at the shortest of delays.

[4]See Harris and Laibson (2013), Grenadier and Wang (2007), Cao and Werning (2018) and Mu et al. (2016) Feldstein (1985), Caliendo and Aadland (2007), Griffin et al. (2012), Hong and Hanna (2014). There are also papers that approach this puzzle by combining behavioral and more traditional factors, such as Campbell and Mankiw (1989) who explain the hump-shaped wages with rule-of-thumb consumers in the economy.

[5]Ericson and Laibson (2019) use the term present focus, rather than the more common term present bias, because they believe the word bias implies a prejudgment that the behavior is a mistake, which is not true in their view.

[6]Present bias, which is viewed as a form of misoptimization that accounts for a range of behavioral "mistakes," e.g. undersaving for retirement, has yielded a large literature that emphasizes the potential for policies like forced pensions or retirement saving subsidies to protect against or correct such mistakes (for a survey on present bias see O'Donoghue and Rabin (2015))

Working in a lifecycle model where households naive about their time-inconsistent preferences repeatedly optimize a logarithmic utility function, we derive conditions on the future weighting factor such that the log consumption profile will be concave. Local concavity at the peak of a hump is a necessary condition for the consumption profile to be hump-shaped.[7] In order for the log consumption profile to be concave at a given point in the lifecycle, the future weighting factor between that point and the end of the life span must be greater on average than the weighting factors at shorter delays, as only these will be relevant going forward. This translates to the discount function decaying at a slower rate than an exponential function over the remaining life span.

In formal terms, suppose that intertemporal preferences from the perspective of period t can be represented by $U_t = \sum_{s=t}^{T} D_{s-t} u_s$, where u_s is the instantaneous utility experienced in period s and D_x reflects the discounting associated with a delay of $x \in \{0, 1, 2, ...\}$. A common example in which the concept of present bias is readily discernible is the β-δ or "quasihyperbolic" functional form

$$D_x = \begin{cases} 1 & \text{if } x = 0 \\ \beta\delta^x & \text{if } x > 0. \end{cases}$$

If $\beta = 1$, this reduces to an exponential discounting function, in which case the optimal plan at $t = 0$ will remain the optimal plan throughout the lifecycle. For $\beta \in (0, 1)$, the utility from consumption at all periods after the present are discounted by the factor β, and the difference $1 - \beta$ is a measure of present bias. The optimal plan at $t = 0$ will differ from the optimal plan later in life as the household will continually seek to advance consumption relative to what she originally planned. Conversely, if $\beta > 1$, the utility from consumption at all future periods would be magnified by the factor β, and $\beta - 1$ can be characterized as a measure of "future bias".

While the quasihyperbolic case only covers a measure-zero subset of the space of all possible discounting functions, because of their simplicity quasihyperbolic discount functions are often used as a proxy for other nonexponential discount functions. Indeed, the terminology of quasihyperbolic derives from this usage as an approximation to hyperbolic discount functions. If $\beta < 1$, the quasihyperbolic discount function will share with hyperbolic dis-

[7] In continuous time, $\frac{d^2 \ln c(t)}{dt^2} = \frac{1}{c(t)} \frac{d^2 c(t)}{dt^2} - \left(\frac{d \ln c(t)}{dt}\right)^2$. A necessary condition for the consumption profile to be concave at t is that the log consumption profile is also concave at t. Since a hump-shaped consumption profile must be locally concave at its maximum, it follows that the log consumption profile must also be locally concave at the maximum.

count functions the property that the lifecycle profile of log consumption is concave. These discount functions also share another property to be further explained below: the household at most ages would prefer the consumption profile it would get if it could commit to its $t = 0$ plan to what it gets in equilibrium after accounting for its changing intertemporal preferences. On the other hand, a future-biased quasihyperbolic function with $\beta > 1$ will yield log consumption profiles that are convex, and the household would usually prefer the consumption profile it actually gets to what it would get if it could commit to its initial plan.

However, as we demonstrate in this paper, the language of "present" and "future" bias are not reliable predictors of these properties. In other words, a concave log consumption profile and almost Pareto dominancy of the consumption profile of the commitment path do not always arise in association with discount functions that one would naturally think of as present-biased. For example, a pure myopic discount function is a discount function that vanishes for delays beyond some horizon. Households with such a discount function do not care about consumption in the future beyond that horizon. Nevertheless, myopia yields properties consistent with a future-biased quasihyperbolic discount function rather than properties consistent with a present-biased quasihyperbolic discount function.[8]

Our approach for exploring the driving force behind the consumption profile begins with an exponential discount function, for which we know there is no time-inconsistency and the log consumption profile will be linear. Measuring the deviation from an exponential discount factor in terms of future weighting factors provides a straightforward way of understanding the origin of a present bias, which comes from having all ε_t be positive and strictly increasing for $t > 1$, or a future bias, which comes from having all ε_t be negative and strictly decreasing for $t > 1$.[9] A present bias is a necessary, albeit not sufficient, condition to have a concave log consumption profile. Likewise, a future bias will be a necessary condition to have a convex log consumption profile.

Positive future weights mean that the discount function will be higher than an exponential discount function as the delay time increases and are necessary for a discount function to exhibit present bias everywhere. Negative future weights mean the opposite and likewise are normally associated with future bias. In the case of a myopic discounting function the ε_t will all be -1 for sufficiently high t, and this will exhibit both present and future bias at different delays. The upshot is that a myopic discount function will tend to behave more like a future-biased quasihyperbolic discount function since they both put <u>less</u> weight on future

[8] A myopic discount function actually exhibits both present and future bias, depending on the time horizon.

[9] For the case of a future bias we also need the additional requirement that $\varepsilon_t > -1$.

consumption relative to an exponential discount function.

Another issue related to present and future bias that has been the focus of a relatively recent literature pertains to welfare analysis. Since an individual with time-inconsistent preferences, whether present- or future-biased, will choose a consumption profile that depends on the time of the choosing, it is not obvious which of these consumption profiles or the preferences at what period of life should be the reference point for welfare comparison. A common solution to this problem in the literature is to use the preferences of the initial self to evaluate welfare (see for example Laibson (1996), Laibson (1997), Laibson (1998), Laibson et al. (1998), O'Donoghue and Rabin (1999), O'Donoghue and Rabin, O'Donoghue and Rabin (2001) among many others). In fact, Caliendo and Findley (2019) show that commitment to the time-zero consumption plan can improve the objective function for all selves if the number of selves exceeds a certain threshold which turned out to be quite small in their setting.

Adding to this strand of literature, the other contribution of this paper is to determine the conditions on future weighting factors under which the commitment path will Pareto dominate the realized path in discrete time.[10] We find that for the time-zero consumption profile to almost[11] Pareto dominate the consumption profiles that are chosen at each period of life we need the future weighting factor at the longest relevant delay, i.e. ε_T, to be sufficiently large. In other words, we find that if we put a heavy weight on the discount factor for the last period of life, as perceived during the first period of life, then at time zero the household will plan to consume a lot in the last period of life. To accomplish this the household will need to save for this terminal consumption at each successive period. However, this desire to consume so much at the end of life is fleeting and disappears for $t > 0$. Thus the household saves more over the course of the commitment path than it does on the realized path, so the later selves would each (except possibly at $t = 1$) prefer the commitment path to the realized path.

When we express the condition for the commitment path of consumption to almost Pareto dominate the realized path in terms of future weighting factors, it is easy to see, for small deviations from an exponential discount function, that this condition comes in the form of an upper bound on the future weighting factor at the longest possible delay. Conversely, the

[10] We only compare the preferences of the households' various selves regarding the commitment path and the realized path. We do not make any claims regarding Pareto efficiency as in Richter (2020), i.e. we do not compare how the various selves value these two paths relative to other feasible consumption paths.

[11] In discrete time, the difference in welfare between the realized and commitment paths at $t = 1$ is closely related to that difference at $t = 0$. Since the commitment path must, by definition, always dominate the realized path at $t = 0$, what happens at $t = 1$ can deviate from what happens over the rest of the life span.

condition for the realized path to almost Pareto dominate the commitment path imposes a lower bound on the future weighting factor at the longest delay. Since the profession usually thinks about policy interventions in favor of the initial path in terms of present bias, this is a very counter-intuitive result. One would imagine that present bias should be determined by the behavior of the discount function at short delays rather than at long delays. In fact, as we demonstrate, that is also not true. Preference reversals for an intertemporal tradeoff that starts t periods in the future will be driven by how the slope of the future weighting factor at a delay of t compares to the slope at its shortest delay. Thus present bias involves the behavior of the discount function at both long and short delays. But whether a household will benefit from socially-imposed commitment mechanisms depends primarily on how slowly the discount function decays relative to an exponential discount function. The condition for almost Pareto dominance of commitment path can be neatly expressed in terms of a sequence of lower bounds on the terminal future weight ε_T that must all be satisfied. This emphasizes that what drives the welfare result is how the household values utility in the distant future rather than the immediate present. Further, we show that to the first order in ε_t and for small life spans, a strictly concave log consumption profile implies almost Pareto dominance of the commitment path.

In this paper we model the household's choices in discrete time. A companion paper, Feigenbaum and Raei (2021), addresses the same issues in continuous time. Aside from the obvious advantages that the majority of economists are most comfortable working in discrete time and that economic data accumulates intermittently rather than continuously, another advantage of working in discrete time is that at each period in the lifecycle the household only has to make a tradeoff between the present and a finite number of future periods. Thus we can isolate the effect of the future weighting factor at every delay, which greatly simplifies the interpretation of our results. In particular, all of our results can be expressed as inequalities relating the future weighting factor at a given delay to the weighting factors at shorter delays.

This paper is organized in the following way. Section 2 describes the model environment including the general format for the discount function. Section 3 develops the condition on the discount function for a concave or convex log consumption profile. Section 4 drives the condition on the discount function under which commitment to the initial plan would almost Pareto dominate the realized plan, and in section 5 we compare this Pareto dominance condition with the condition for concavity/convexity of the log consumption profile. Section 6 concludes.

2 Model environment

We focus on a finite-horizon life-cycle model in which households live for $T+1$ periods. The household earns income $y_t \geq 0$ at age t for $t = 0, ..., T$, which can be consumed c_t or saved as k_{t+1} at a fixed gross interest rate $R \geq 0$. [12]

2.1 Household optimization problem

At time t, a household with existing saving k_t maximizes

$$U_t = \sum_{s=t}^{T} D_{s-t} \ln c_{s|t}$$

subject to

$$c_{s|t} + k_{s+1|t} = y_s + Rk_{s|t}, \quad s = t, \ldots, T,$$

where $D_t \geq 0$ is the discount function, and $c_{s|t}$ and $k_{s+1|t}$ are consumption and saving at period s as planned in period t.[13] We will normalize $D_0 = 1$ and will also assume that $D_1 > 0$. The latter condition ensures that it will never be optimal for the household to consume all of its remaining wealth in the present period, which would leave the future selves with utility of $-\infty$. Note that the household will solve this problem with $k_{t|t} = k_t$ and $k_{T+1|t} = 0$. To simplify notation, we will assume the household begins with $k_0 = 0$.[14]

Let us define

$$h_t = \sum_{s=t}^{T} \frac{y_s}{R^{s-t}}, \tag{1}$$

[12] It is worth mentioning that similar to Drouhin (2020), in this paper we use the "choice-based" methodology which compares the solutions of dynamic programs with different decision dates. It is the methodology used originally by Strotz (1956), and now standard in behavioral macroeconomics, since the pioneering work of Laibson (1994), Laibson (1997), O'Donoghue and Rabin (1999). It is "choice based" because it not only uses a utility function that represents the preference relation, but also the budgetary constraints that the decision maker faces.

[13] The results are not qualitatively different for other CRRA utility functions, but they are more complicated so we only consider the logarithmic case. In solving the model we will proceed as though the household is naive about its time-inconsistency and does not know it will revise its plans as its preferences change. We could alternatively assume that the household is sophisticated about its time-inconsistency. However, with logarithmic period utility, the realized path (and the commitment path in Section 4) will be the same under both assumptions, so there is no loss of generality between naivete and sophistication in the results documented here. For more discussion see Marin-Solano and Navas (2009).

[14] Our results easily generalize if the household is endowed with savings or debt at birth.

which represents the present value of the income stream from period t onward. Note that

$$h_t = y_t + \sum_{s=t+1}^{T} \frac{y_s}{R^{s-t}} = y_t + \frac{h_{t+1}}{R} \tag{2}$$

for $t < T$. We can combine the period budget constraints from t to T into a lifetime budget constraint as of t:

$$\sum_{s=t}^{T} \frac{c_{s|t} + k_{s+1|t}}{R^{s-t}} = \sum_{s=t}^{T} \frac{y_s + Rk_{s|t}}{R^{s-t}}.$$

Using (1) and (2), this simplifies to

$$\sum_{s=t}^{T} \frac{c_{s|t}}{R^{s-t}} = h_t + Rk_t \tag{3}$$

The Lagrangian of the household problem at t can then be written as

$$L_t = \sum_{s=t}^{T} \left[D_{s-t} \ln c_{s|t} - \frac{\lambda_t c_{s|t}}{R^{s-t}} \right] + \lambda_t [h_t + Rk_t]. \tag{4}$$

Therefore, the first order condition (FOC) with respect to consumption will be

$$\frac{\partial L_t}{\partial c_{s|t}} = \frac{D_{s-t}}{c_{s|t}} - \frac{\lambda_t}{R^{s-t}} = 0. \tag{5}$$

The initial consumption plan $c_{s|0}$ that is determined at $t = 0$, the first period of life, will be referred to hereafter as the **commitment path**. Note, however, that unless the discount function is exponential the household will only follow the initial plan at $t = 0$. Indeed, at each period t of life, the household will choose a new plan $c_{s|t}$, but only the choice of consumption at t, $c_t = c_{t|t}$, will adhere to this plan. As the household progresses from period to period, its preferences will unexpectedly change since we are assuming that the household is naive about the change in its future preferences. When it gets to $t + 1$, it will then have saving $k_{t+1} = k_{t+1|t}$, but it will solve (4) anew, updated to $t + 1$. The resulting consumption profile c_t, determined at each period t, will be referred to as the **realized path**.

While the FOC (5) for $t = 0$ governs the whole commitment path for consumption $c_{s|0}$ from $s = 0, \ldots, T$, along the realized path only the FOC with $s = t$ will actually matter.

This simplifies to

$$\frac{D_{t-t}}{c_{t|t}} - \frac{\lambda_t}{R^{t-t}} = 0,$$

so we have

$$\lambda_t = \frac{1}{c_t}$$

since $c_t = c_{t|t}$ and $D_0 = 1$. The future plan $c_{s|t}$ at t is only relevant to the extent that it determines the Lagrange multiplier λ_t. Generalizing (5), we obtain

$$c_{s|t} = \frac{D_{s-t}R^{s-t}}{\lambda_t} = D_{s-t}R^{s-t}c_t.$$

Inserting these into the lifetime budget constraint (3), we get

$$\sum_{s=t}^{T} \frac{D_{s-t}R^{s-t}c_t}{R^{s-t}} = h_t + Rk_t,$$

which reduces to

$$c_t = \frac{h_t + Rk_t}{\sum_{s=t}^{T} D_{s-t}}. \tag{6}$$

Hence, on the realized path, the budget constraint on period t can be written as

$$k_{t+1} = k_{t+1|t} = y_t + Rk_t - c_t = y_t + Rk_t - \frac{h_t + Rk_t}{\sum_{s=t}^{T} D_{s-t}}. \tag{7}$$

We can use this to calculate an effective Euler equation along the realized path. Combining (2) and (7), we get,

$$\begin{aligned} h_{t+1} + Rk_{t+1} &= R\left(\frac{h_{t+1}}{R} + y_t + Rk_t - \frac{h_t + Rk_t}{\sum_{s=t}^{T} D_{s-t}}\right) \\ &= R\left(h_t + Rk_t - \frac{h_t + Rk_t}{\sum_{s=t}^{T} D_{s-t}}\right) \\ &= R\left(\frac{\sum_{s'=t+1}^{T} D_{s'-t}}{\sum_{s=t}^{T} D_{s-t}}\right)(h_t + Rk_t). \end{aligned}$$

Updating (6) to $t+1$, consumption at $t+1$ is

$$c_{t+1} = R\left(\frac{\sum_{s'=t+1}^{T} D_{s'-t}}{\sum_{s=t}^{T} D_{s-t}}\right)\frac{h_t + Rk_t}{\sum_{s=t+1}^{T} D_{s-t-1}}$$

Applying (6) again in its original form, the Euler equation in equilibrium for a general discounting function D_t with log utility is

$$c_{t+1} = R\frac{\sum_{s'=t+1}^{T} D_{s'-t}}{\sum_{s=t+1}^{T} D_{s-t-1}}c_t. \tag{8}$$

As mentioned above, since $D_1 > 0$, c_{t+1} will be strictly positive.

In the special case of an exponential discount function $D_t = \delta^t$, the ratio

$$\mathcal{D}_t = \frac{\sum_{s'=t+1}^{T} D_{s'-t}}{\sum_{s=t+1}^{T} D_{s-t-1}}$$

simplifies to the constant δ, and we get back the familiar Euler equation $c_{t+1} = \delta R c_t$. More generally, though, for a nonexponential discount function, the inverse ratio $\mathcal{D}_t^{-1}$ measures the gross rate of change in the sum of the discount functions relevant at periods $t+1$ to T as the household moves from t to $t+1$. That is to say the change from the sum $D_1 + \cdots + D_{T-t}$ applicable at t to the sum $1 + \cdots + D_{T-t-1}$ applicable at $t+1$. The richer consumption dynamics that can be obtained in equilibrium with nonexponential discounting functions stems entirely from the deviation of the $\mathcal{D}_t$ from a constant, which will depend on how the discount function D_t deviates from an exponential function.

2.2 Future Weighting Discount Function

Given a discount function $D_t \geq 0$ for $t = 0, ..., T$, we define the "future weighting factor" ε_t via

$$D_t = D_1^t(1 + \varepsilon_t), \tag{9}$$

where D_1 is the discount factor for one period ahead. This future weighting factor basically captures the extra (or diminished, if negative) weight that we put on the discounting t periods in the future. Since we normalize $D_0 = 1$, by definition we will have $\varepsilon_0 = \varepsilon_1 = 0$. Note that this general form of discounting function can accommodate the standard geometric discounter, for which $D_t = \delta^t$, quasi-hyperbolic agents, for whom $D_t = \beta\delta^t$ where

$\beta < 1$ (Laibson (1997)), future-biased agents for whom $D_t = \beta\delta^t$ with $\beta > 1$; and the immediate successor agents, for whom $D_1 = \delta$ and $D_2 = D_3 = \cdots = 0$ (see, Lane and Mitra (1981), Leininger (1986) and Bernheim and Ray (1987)). For example, we can represent a quasihyperbolic discount function by setting $\varepsilon_t = \beta^{1-t} - 1$;

$$D_t = \beta\delta^t$$

for $t > 0$ with $D_0 = 1$. Since

$$D_1 = \beta\delta,$$

ε_t can be calculated as

$$\frac{D_t}{D_1^t} = \frac{\beta\delta^t}{\beta^t\delta^t} = \beta^{1-t} = 1 + \varepsilon_t.$$

Hence

$$\varepsilon_t = \beta^{1-t} - 1. \tag{10}$$

Likewise, for a myopic discounting function that vanishes for $t \geq t^*$, we have $\varepsilon_t = -1$ for $t \geq t^*$.

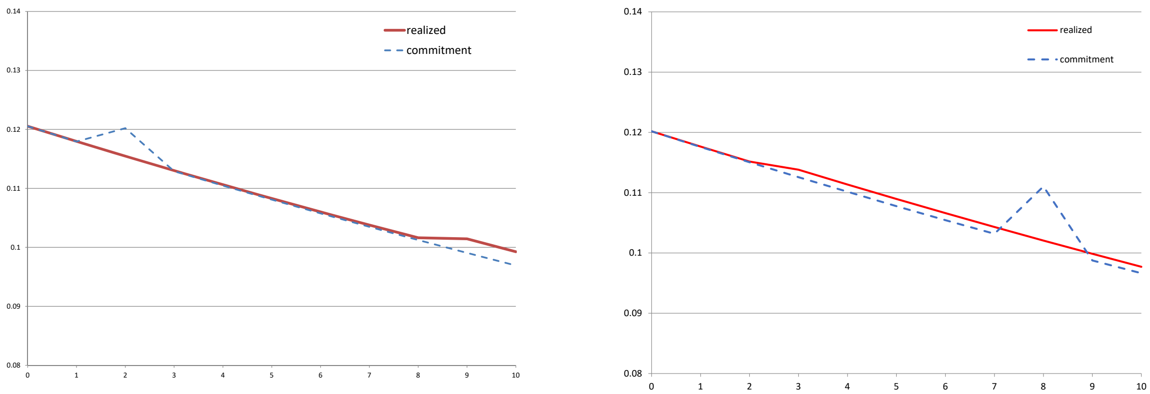
Note that if $\varepsilon_t = 0$ for all t, the discount function will be exponential. Thus we can think of the future weighting factor, ε_t, as the parameter that measures the discount function's deviation from an exponential at the delay t. If $\varepsilon_t > 0$, the discount factor of t periods in the future will be higher than an exponential discount factor. This means utility from consumption t periods in the future will be weighted more heavily than it would be under an exponential discount function. Conversely, with $\varepsilon_t < 0$ the weight on utility consumption t periods in the future will be lower relative to an exponential discount function.[15]

To have a better understanding of the role of ε_t in determining consumption behavior, figure 1 compares the consumption profile under the commitment path and the realized path for a ten period model, $T = 10$. We consider two cases to demonstrate the role of an individual ε_t. First, we have a discount function for which ε_t is zero for all t except $t = 2$. Second, we have a discount function for which ε_t is zero for all t except $t = 8$.

In both plots, the blue dashed line shows the commitment path and the red solid line shows the realized path. In figure 1a, we see a spike in period two along the commitment path simply because $\varepsilon_2 > 0$ means that the household initially puts a higher weight on the utility from consuming two periods ahead compared to all other future periods. Hence, the

[15] To be very precise, as we have defined the future weighting factor, we are talking about a departure from exponential discounting at the rate used between period 0 and 1. In discrete time, it is natural to think of the deviation of D_t from D_1^t, and this will yield some helpful simplifications.

Figure 1: consumption profile, commitment path and realized path



a . only $\varepsilon_2 > 0$

b . only $\varepsilon_8 > 0$

Note: on both graphs the horizontal axis is time and vertical access is the consumption level at each period.

spike at $t = 2$. Likewise, looking at figure 1b in which $\varepsilon_8 > 0$, the spike in the commitment path is at $t = 8$.

The effect of ε_t on the realized path is much more subtle than for the commitment path. With $\varepsilon_2 > 0$, shown in figure 1a, the household continually plans to have high consumption two periods ahead, as happens at $t = 2$ on the commitment path. However, with each new period, she reoptimizes and pushes forward when she intends to have high consumption. This trend continues until the household arrives at period nine of her lifetime, at which point there no longer is a period two periods ahead. Consequently, the realized consumption path is quite smooth, as it would be with exponential discounting, for $t < 9$. She does not realize this intended high consumption two periods ahead until she can no longer defer this consumption. From this point, all future periods are discounted with the same rate. Consumption jumps up in these last two periods as she finally consumes the saving she accumulated to finance the planned extra consumption two periods ahead.

The same intuition applies to figure 1b in which $\varepsilon_8 > 0$. There, the future period with a higher discounting factor disappears after the second period. That is the reason why the realized consumption plan for $t \geq 3$ shifts upward. The high ε_8 disappears from her calculus once there no longer is a period eight periods ahead within her remaining time horizon. Consequently, she behaves like an exponential discounter thereafter, smoothing out over all the periods with $t \geq 3$ the extra consumption that she had previously intended, at $t = 2$, to save entirely for the last period.

The consumption-hump literature has traditionally characterized the effect of the discount function on the shape of the (log) consumption profile in terms of present bias. By examining present and future bias in terms of future weighting factors, we can also get some new insight into the origin of these concepts. A discount function exhibits present bias at $t > 0$ if it gives rise to the following type of preference reversal. Suppose for some allocation $\{c_t\}_{t=0}^{T}$, there exists $\xi_t > 0$ and $\xi_{t+1} \in (0, c_{t+1})$ such that the household would prefer at time 0 the original allocation over a forward-shifted allocation with c_t increased by ξ_t and c_{t+1} decreased by ξ_{t+1}. However, when the household gets to time t, it in stead prefers the forward-shifted allocation over the original allocation. Thus the household would prefer not to shift consumption forward when the possibility of doing so is in the future, but it would opt to make that shift in the present. This is usually interpreted as the household putting an extra preference on consumption in the immediate present. Future bias at $t > 0$ is defined similarly except the preference reversal goes the other way. The household would prefer the forward-shifted allocation over the original allocation when t is in the future, and prefers the original allocation when it reaches time t. We say a discount function is present-biased (future-biased) if it exhibits present (future) bias at all $t > 0$.

Assuming $D_s > 0$ for all s, we can express the condition for preference reversals in terms of the perceived marginal rate of substitution between consumption at t and consumption at $t + 1$ as of time $s \leq t$:

$$m_s(t) = \frac{D_{t+1-s} u'(c_{t+1})}{D_{t-s} u'(c_t)}.$$

The household will prefer the forward-shifted allocation at time 0 and the original allocation at t if

$$D_t u'(c_t)\xi_t - D_{t+1} u'(c_{t+1})\xi_{t+1} < 0 < u'(c_t)\xi_t - D_1 u'(c_{t+1})\xi_{t+1},$$

which we can rearrange as

$$m_0(t) = \frac{D_1 u'(c_{t+1})}{u'(c_t)} < \frac{\xi_t}{\xi_{t+1}} < \frac{D_{t+1} u'(c_{t+1})}{D_t u'(c_t)} = m_t(t).$$

The household will have a present bias at t if $m_0(t) < m_t(t)$ since we can then find ξ_t and ξ_{t+1} such that $\frac{\xi_t}{\xi_{t+1}} \in (m_0(t), m_t(t))$. Since $\varepsilon_1 = 0$ by definition, this reduces to the condition

$$1 < \frac{1 + \varepsilon_{t+1}}{1 + \varepsilon_t},$$

or equivalently

$$\varepsilon_t < \varepsilon_{t+1}.$$

It will be helpful in the following to define the future weighting growth factor

$$\phi_t = \frac{1+\varepsilon_{t+1}}{1+\varepsilon_t} \tag{11}$$

at t, assuming $\varepsilon_t > -1$, since many of our results depend on such ratios. Note that we have the theorem that $\phi_t \gtreqless 1$ if and only if $\varepsilon_{t+1} \gtreqless \varepsilon_t$.

Thus a present-biased discount function will have strictly increasing and positive (for $t > 2$) future weighting factors. Conversely, a strictly positive and future-biased discount function will have strictly decreasing and negative (for $t > 2$) future weighting factors.[16] To put this in more graphical terms, a present-biased discount function will lie above the exponential function defined by the discounting between time delay 0 and time delay 1, and the divergence between the curves must increase with the time delay. A future-biased discount function will lie below the same exponential function, and the divergence between the curves must also increase (while avoiding zero as we discuss below). This is counterintuitive because one would think a present bias would be determined by the behavior of the discount function at short delays when in fact it depends on the behavior of the discount function at all and particularly the longest delays.

Note that a myopic discount function that is zero for t greater than equal to some $t^* > 1$ does not fit nicely into the categories of a present- or future-biased discount function because it does not satisfy the caveat that the D_t are all positive, which is necessary for the marginal rate of substitution between c_t and c_{t+1} to be defined. There will be a future bias at $t^* - 1$ since at time zero the household would prefer not to consume anything at t^*, but its $(t^* - 1)$-utility is only defined if $c_{t^*} > 0$. On the other hand, there will be a weak present bias at $t \geq t^*$ since at time zero the household will be indifferent between how it allocates consumption between t and $t + 1$. However, at time t the household will prefer to have more consumption at t.

[16]A related property of discount functions is increasing patience (Prelec (2004)). Since Prelec defines this concept in continuous time, we refer the reader to our companion paper in continuous time, Feigenbaum and Raei (2021), for an understanding of how it translates into a property of the future weighting factors.

3 Curvature of the log consumption profile

Empirically, lifecycle profiles of household consumption are hump-shaped, and time-inconsistency is often invoked as an explanation for this phenomenon. As we discussed in the previous section, ε_t is the parameter that controls the discounting weight of future periods. In this section, we explore how the value of the future weighting factor, ε_t, determines the curvature of the log consumption profile of the household. More precisely, we establish a necessary condition on ε_t under which the log consumption profile would be locally concave (convex) at age $T-t$. This in turn is a necessary condition for the consumption profile to have a local maximum at age $T-t$. [17]

As a first step, we will rewrite the Euler equation in terms of the future weighting discount function. Replacing the general form of discounting function D_t in the household's Euler equation (8) with the form involving the future weighting discounting function (9) gives us

$$c_{t+1} = D_1 R \frac{\sum_{s'=t+1}^{T} D_1^{s'}(1+\varepsilon_{s'-t})}{\sum_{s=t+1}^{T} D_1^{s}(1+\varepsilon_{s-t-1})} c_t. \tag{12}$$

In this still exact form, it is more apparent that the Euler equation reduces to the usual $c_{t+1} = D_1 R c_t$ when we have an exponential discounting function and $\varepsilon_2 = \varepsilon_3 = \cdots = \varepsilon_T = 0$. Alternatively, by setting $z = s - t$, we can rewrite this exact Euler equation (12) as

$$\frac{c_{t+1}}{c_t} = D_1 R \frac{\sum_{z'=1}^{T-t} D_1^{z'}(1+\varepsilon_{z'})}{\sum_{z=1}^{T-t} D_1^{z}(1+\varepsilon_{z-1})}. \tag{13}$$

As we will often do in the following, it is helpful to consider how this equation simplifies in the limit of small future weighting factors. Since the zeroth-order terms that do not involve the ε_z are the same in the numerator and the denominator of (13), we can rearrange the equation to obtain

$$\frac{c_{t+1}}{c_t} = D_1 R \frac{1 + \frac{\sum_{z'=1}^{T-t} D_1^{z'} \varepsilon_{z'}}{\sum_{s'=1}^{T-t} D_1^{s'}}}{1 + \frac{\sum_{z=1}^{T-t} D_1^{z} \varepsilon_{z-1}}{\sum_{s=1}^{T-t} D_1^{s}}}.$$

[17] If the consumption profile has a local maximum at $t^a st$, it will, of course, also be necessary to have $\frac{c_{t^a st}}{c_{t^a st-1}} > 1 > \frac{c_{t^a st+1}}{c_{t^a st}}$. However, the main hurdle is constructing a model where the growth rate of consumption changes. Adjusting the model so we quantitatively get growth rates both above and below 1 is a matter of calibration. In a partial-equilibrium environment where R is a free parameter, this is trivial. In a general-equilibrium environment, it is more challenging but still less of an issue than getting a concave profile in the first place.

We define the difference operator Δ such that for a time series x_t we have

$$\Delta x_t = x_{t+1} - x_t. \tag{14}$$

To first order in the ε, the Euler equation then approximates to

$$\frac{c_{t+1}}{c_t} = D_1 R \left[1 + \frac{\sum_{z=0}^{T-t-1} D_1^z \Delta\varepsilon_z}{\sum_{s=0}^{T-t-1} D_1^s} \right] + O(\varepsilon^2), \tag{15}$$

where $O(g(x))$ represents an unspecified function smaller than $Mg(x)$ for some $M > 0$ in the limit as $x \to 0$.

Eq. (15) shows that deviations of the Euler equation from the canonical Euler equation $c_{t+1} = D_1 R c_t$ for an exponential discounting function arise because of changes in the future weighting as the delay changes by one period. The effect of a change in future weighting s periods in the future is discounted by D_1^s, so a change in the future weighting at short delays will have a bigger effect than a change at long delays.

We will now focus on the log consumption profile, which will be concave if $\log(\frac{c_{t+1}}{c_t})$ decreases with t. We can take logs of both sides of equation (13) and difference it to obtain

$$\Delta \ln c_t = \ln(D_1 R) + \ln\left(\frac{\sum_{s'=1}^{T-t} D_1^{s'}(1+\varepsilon_{s'})}{\sum_{s=1}^{T-t} D_1^s(1+\varepsilon_{s-1})} \right). \tag{16}$$

Similarly, we can define the second-order difference

$$\Delta^2 \ln c_t = \ln\left(\frac{\sum_{z'=1}^{T-t-1} D_1^{z'}(1+\varepsilon_{z'})}{\sum_{z=1}^{T-t-1} D_1^z(1+\varepsilon_{z-1})} \right) - \ln\left(\frac{\sum_{s'=1}^{T-t} D_1^{s'}(1+\varepsilon_{s'})}{\sum_{s=1}^{T-t} D_1^s(1+\varepsilon_{s-1})} \right),$$

which simplifies to

$$\Delta^2 \ln c_t = \ln\left(\frac{\sum_{z'=1}^{T-t-1} D_1^{z'}(1+\varepsilon_{z'})}{\sum_{s'=1}^{T-t} D_1^{s'}(1+\varepsilon_{s'})} \frac{\sum_{s=1}^{T-t} D_1^s(1+\varepsilon_{s-1})}{\sum_{z=1}^{T-t-1} D_1^z(1+\varepsilon_{z-1})} \right). \tag{17}$$

The log consumption profile will be concave iff $\Delta^2 \ln c_t \leq 0$ for $t = 0, ..., T-2$. If $\Delta^2 \ln c_t < 0$ for all $t = 0, ..., T-2$, then the log consumption profile will be strictly concave. The reverse inequalities will yield convex and strictly convex profiles.[18]

[18]Unlike in continuous time, for the log consumption profile to be strictly concave (convex) at $t+1$ we must have $\Delta^2 \ln c_t$ be negative (positive). If the second difference vanishes, the profile must be locally linear.

Notice that the $\ln D_1 R$ in (16) vanishes from (17). Absent the future weighting factors in (17), the argument of the logarithm is clearly one, so all of the surviving terms on the right-hand side are of first or higher order in the ε_t, corroborating again that the log consumption profile with an exponential discounting function is exactly linear. Any deviation from linearity is driven by the future weighting factors.

Consequently, if the log consumption is concave at $t+1$, we must have

$$\frac{\sum_{z'=1}^{T-t-1} D_1^{z'}(1+\varepsilon_{z'})}{\sum_{s'=1}^{T-t} D_1^{s'}(1+\varepsilon_{s'})} \frac{\sum_{s=1}^{T-t} D_1^{s}(1+\varepsilon_{s-1})}{\sum_{z=1}^{T-t-1} D_1^{z}(1+\varepsilon_{z-1})} \leq 1, \tag{18}$$

which can be simplified to[19]

$$\varepsilon_{T-t} \geq \frac{\sum_{s'=0}^{T-t-2} D_1^{s'}(1+\varepsilon_{s'+1})}{\sum_{s=0}^{T-t-2} D_1^{s}(1+\varepsilon_{s})}(1+\varepsilon_{T-t-1}) - 1. \tag{19}$$

Note that all of the future weighting factors on the right-hand side are at delays shorter than $T-t$. Thus the exact condition for concavity at $t+1$ is a lower bound on ε_{T-t} that depends on future weighting factors at shorter delays.

If $\varepsilon_{T-t-1} = -1$, so $D_{T-t-1} = 0$, there are two possibilities in terms of the shape of the log consumption profile at $t+1$. These depend on ε_{T-t}. If $\varepsilon_{T-t} = -1$ too, then $\Delta \ln c_t = \Delta \ln c_{t+1}$, and the log consumption profile will be linear (and thus both weakly concave and weakly convex) in the vicinity of $t+1$. If, on the other hand, $\varepsilon_{T-t} > -1$, $\Delta \ln c_t > \Delta \ln c_{t+1}$, and the log consumption profile will be strictly concave in the vicinity of $t+1$.

Proposition 1. *If $\varepsilon_s > -1$ for all $s = 0, \ldots, T-t-1$, the log consumption profile will be strictly concave locally at $t+1$ iff*

$$\phi_{T-t-1} > \frac{\sum_{s=0}^{T-t-2} D_s \phi_s}{\sum_{s'=0}^{T-t-2} D_{s'}}. \tag{20}$$

The profile will be strictly convex locally if the inequality is reversed.

This follows from (19) using (11).

So the concavity condition at $t+1$ is that ϕ_{T-t-1} is bigger than a weighted average of the ϕ_s for $s = 0, ..., T-t-2$, where the weights are the D_s. That is to say, the log consumption profile will be concave when there are s periods remaining if and only if the future weighting

[19]See appendix A for details on this calculation.

growth factor at s is bigger than a weighted average of the future weighting growth factor at shorter delays.

To first order in the future weighting factors, the future weighting growth factor approximates to

$$\phi_t = \Delta\varepsilon_t + O(\varepsilon^2). \tag{21}$$

This implies that to first order, concavity of log consumption at t requires

$$\Delta\varepsilon_{T-t} \geq \frac{\sum_{i=0}^{T-t-1} D_1^i \Delta\varepsilon_i}{\sum_{i'=0}^{T-t-1} D_1^{i'}}. \tag{22}$$

Since $\Delta\varepsilon_0 = \varepsilon_1 - \varepsilon_0 = 0$ by definition, this condition for concavity at $T-1$ is $\varepsilon_2 = \Delta\varepsilon_1 > 0$. Note also that the exact condition (19) for concavity at $T-1$ is the same. A positive future weighting at a delay of two periods means consumption growth between $T-1$ and T will be lower than between $T-2$ and $T-1$, producing a concavity in the log consumption profile at the end of life. Conversely, a negative future weighting will result in a convex log consumption profile between $T-2$ and T.

Since a weighted average of a heterogeneous set must be less than the maximum in the set and greater than the minimum in the set, it follows immediately from Proposition 1 that if the ϕ_t are strictly increasing (decreasing) then the log consumption profile will be strictly concave (convex). Moreover, if the ϕ_t are increasing with $\phi_1 > 1$, the log consumption profile will be strictly concave. Likewise, if the ϕ_t are decreasing with $\phi_1 < 1$, the log consumption profile will be strictly convex.

Proposition 2. *For the entire log consumption profile to be strictly concave (convex), the $\Delta\varepsilon_s$ from $s = 1, \ldots, T-1$ must all be positive (negative) and the ϕ_s must all be greater (less) than 1. Consequently, the ε_s from $s = 2, \ldots, T$ must all be strictly increasing (decreasing). In other words, a necessary condition for the log consumption profile to be strictly concave (convex) is that the discount function is present-biased (future-biased).*

This proposition can be proved by induction. Suppose the $\phi_i > 1$ for $s = 1, \ldots, s-1$. Then (20) implies $\phi_s > 1$, and $\varepsilon_{s+1} = \varepsilon_s + \Delta\varepsilon_s > \varepsilon_s > 0$. Note also that each successive iteration of (20) is the necessary condition for the log consumption profile to be concave one period earlier. Thus the condition that

$$\varepsilon_T > \frac{1}{D_1} \frac{\sum_{s'=1}^{T-1} D_1^{s'}(1+\varepsilon_{s'})}{\sum_{s=0}^{T-2} D_1^s(1+\varepsilon_s)}(1+\varepsilon_{T-1}) - 1.$$

is the condition that the log consumption profile is strictly concave between $t = 0$ and $t = 2$. Iterating forward in time, each log consumption growth ratio will depend on one more difference $\Delta\varepsilon_s$ than the ensuing log consumption growth ratio, so $\Delta\varepsilon_s > 0$, or equivalently $\varepsilon_{s+1} > \varepsilon_s$ will be necessary to have the log consumption growth ratio decrease with time. One way to think about this result is that what often gets referred to as present bias is really a case of young households putting extra weight on consumption in the distant future. However, consumption in these future periods gradually matters less to the household as the future gets closer to the present. Therefore, the ε_t must grow with t because that implies the extra weight associated with a specific age gets smaller as we approach that age and the delay time gets shorter.

How fast must the future discounting weights grow to get a strictly concave log consumption profile? We can get a fairly simple idea of this using the first-order approximation (22). For a given ε_2, let us define a lower bound, $\Delta\underline{\varepsilon}_s$, on $\Delta\varepsilon_s$ such that the corresponding (22) holds with equality. For the case of $s = 1$, (23) evaluates straightforwardly to

$$\Delta\underline{\varepsilon}_2 = \frac{D_1}{1 + D_1}\varepsilon_2.$$

To first order, a necessary and sufficient condition for strict concavity from $t = T - 3$ to $t = T$ is that $\varepsilon_2 = \Delta\varepsilon_1 > 0 = \Delta\underline{\varepsilon}_1$ and $\Delta\varepsilon_2 > \Delta\underline{\varepsilon}_2$. We can thus iteratively define

$$\Delta\underline{\varepsilon}_{s+1} = \frac{\sum_{i=0}^{s} D_1^i \Delta\underline{\varepsilon}_i}{\sum_{i'=0}^{s} D_1^{i'}} = \frac{\sum_{i=1}^{s} D_1^i \Delta\underline{\varepsilon}_i}{\sum_{i'=0}^{s} D_1^{i'}}, \tag{23}$$

where $\Delta\underline{\varepsilon}_1 = \varepsilon_2$ is given. If $\Delta\varepsilon_i > \Delta\underline{\varepsilon}_i$ for $i = 2, \ldots, s$ are all necessary conditions for strict concavity, then $\Delta\varepsilon_{s+1} > \Delta\underline{\varepsilon}_{s+1}$ will also be a necessary condition for strict concavity. To put it another way, if the $\Delta\varepsilon_s = \Delta\underline{\varepsilon}_s$ for $s = 2, \ldots, T - 1$ and $\varepsilon_2 > 0$, the log consumption profile will be linear from $t = 0$ to $t = T - 1$ and strictly concave between $t = T - 2$ and $t = T$.

Proposition 3. *For $t \geq 2$,*

$$\Delta\underline{\varepsilon}_t = \frac{D_1}{1 + D_1}\varepsilon_2.$$

The proof follows by induction. If it is true for $2, \ldots, s$,

$$\begin{aligned} \Delta\underline{\varepsilon}_{s+1} &= \frac{\sum_{i=1}^{s} D_1^i \Delta\underline{\varepsilon}_i}{\sum_{i'=0}^{s} D_1^{i'}} = \frac{D_1\varepsilon_2 + \sum_{i=2}^{s} D_1^i \Delta\underline{\varepsilon}_2}{\sum_{i'=0}^{s} D_1^{i'}} \\ &= \frac{D_1 + \frac{D_1}{1+D_1}\sum_{i=2}^{s} D_1^i}{\sum_{i'=0}^{s} D_1^{i'}}\varepsilon_2 \\ &= \frac{D_1}{1+D_1}\frac{\sum_{i=0}^{s} D_1^i}{\sum_{i'=0}^{s} D_1^{i'}}\varepsilon_2 = \frac{D_1}{1+D_1}\varepsilon_2. \end{aligned}$$

Therefore, a necessary condition for the log consumption profile to be concave is to have $\varepsilon_2 > 0$ and

$$\Delta\varepsilon_s \geq \Delta\underline{\varepsilon}_s = \frac{D_1}{1+D_1}\varepsilon_2$$

for $s = 2, \ldots, T-1$. These results are only approximate, but they imply that weak concavity requires the ε_t to grow, albeit they need only grow linearly with a slope greater than $\frac{D_1}{1+D_1}\varepsilon_2$. Note however that, if the future discounting weights grow faster than linearly at short delays, they must continue to grow faster than linearly at longer delays to maintain strict concavity over the whole lifespan.

It is well known that the quasihyperbolic and hyperbolic discount functions yield concave log consumption profiles. This follows from Proposition (1) since, in the hyperbolic case, the ϕ_t are strictly increasing and, in the quasihyperbolic case, the ϕ_t are constant and greater than 1 for $t > 0$. For both of these discount functions $\Delta\varepsilon_s$ is also strictly increasing for $s \geq 1$, so the first-order approximation to the concavity condition (22) is also satisfied.

4 Pareto dominance of the commitment path

In previous sections, we described the household problem with a discounting function that depends on the time to consumption from the present rather than the absolute time when the consumption occurs. Such a household has time-inconsistent preferences and therefore, as Strotz (1955a) noted, the marginal rate of substitution between consumption at different times depends on when the household is evaluating the utility from these consumptions. Consequently, the household at different ages will value consumption plans differently. This multiplicity of selves can substantially complicate welfare analysis.

A common solution to tackle this complication in the literature is to use the preferences of the initial self to evaluate welfare. See, for example, (Laibson, 1997, 1996), Laibson et al. (1998), and (O'Donoghue and Rabin, 1999, 2001). This approach does have its criticisms however. Dewatripont et al. (2004) states that there is "no normative foundation" for equating welfare with time-zero preferences.

A more recent literature explores conditions under which committing to the initial plan of the time-zero self improves the welfare of all selves over the life cycle as compared to what they would actually obtain over the lifecycle, providing a justification for singling out the preferences of the time-zero self. Caliendo and Findley (2019) show that with quasihyperbolic discounting commitment to the time-zero consumption plan can improve the objective function for all selves if the number of selves exceeds a certain threshold which turned out to be quite small in their setting. Expanding upon this result, Feigenbaum and Raei (2021) show in a continuous-time setup conditions under which commitment to the initial plan will be Pareto improving for all the different selves. However, because almost all of these conditions involve integrals of future weighting factors instead of sums, they are difficult to interpret. In this section we obtain similar but simpler results in discrete time, formulating a condition on the future weighting factors under which committing to the initial plan will almost Pareto dominate the realized plan.

The realized utility as of time t is simply the realized value of the household's objective function at time t:

$$U_t^* = \sum_{s=t}^{T} D_{s-t} \ln(c_s).$$

In contrast, the commitment utility at time t is

$$U_t^c = \sum_{s=t}^{T} D_{s-t} \ln(c_{s|0}), \tag{24}$$

which is what you obtain if you insert the original $t = 0$ consumption path into the objective function at time t. What concerns us most is the difference ΔU_t in realized utility between the realized plan and the original plan at time t:

$$\Delta U_t = U_t^* - U_t^c = \sum_{s=t}^{T} D_{s-t} \ln\left(\frac{c_s}{c_{s|0}}\right). \tag{25}$$

Note that if $\Delta U_t > 0$, then following the realized consumption plan provides the household with a higher utility compared to the initial plan. Conversely, if $\Delta U_t < 0$, then committing to the initial plan is optimal. This is a general form and D_{s-t} can be replaced with any discounting function. For example with $D_t = \beta^t$ both the original plan and realized plan coincide and $\Delta U_t = 0$, meaning that the household will be indifferent between the two. By definition, the commitment path must maximize lifetime utility at $t = 0$, so we must have $\Delta U_0 \leq 0$. In what follows, we will see that $\Delta U_1 = 0$ must always hold to first order in the future weighting factors. We will then investigate conditions on the future weighting factors under which the initial path would almost Pareto dominate the realized path for the household to first order, i.e. $\Delta U_t < O(\varepsilon^2)$ for $t > 1$.

For the original plan $c_{t|0}$, the consumption at period t, as determined at period 0, can be written

$$c_{t|0} = D_t R^t c_0 = D_1^t (1 + \varepsilon_t) R^t c_0.$$

Let us define

$$c_t^0 = D_1^t R^t c_0.$$

Therefore

$$\ln c_{t|0} = \ln c_t^0 + \varepsilon_t + O(\varepsilon^2), \tag{26}$$

and we can simplify the utility obtained at time t from committing to the initial consumption plan as

$$U_t^c = \sum_{s=t}^{T} \left[D_{s-t} \ln(c_s^0) + D_1^{s-t} \varepsilon_s \right] + O(\varepsilon^2). \tag{27}$$

To compute the realized consumption allocation, we must work with the effective Euler equation of the household problem. As we showed in section 3, this effective Euler equation is (15) to first order in the ε_t. Thus

$$\ln c_t = \ln(D_1 R) + \frac{\sum_{s=t}^{T} D_1^s \Delta\varepsilon_{s-t}}{\sum_{s'=t}^{T} D_1^{s'}} + \ln c_{t-1} + O(\varepsilon^2) \tag{28}$$

Iterating (28) from $\ln c_0$ to $\ln c_t$, we get

$$\ln c_t = t \ln D_1 R + \ln c_0 + \sum_{i=1}^{t} \frac{\sum_{s=i}^{T} D_1^s \Delta\varepsilon_{s-i}}{\sum_{s'=i}^{T} D_1^{s'}} + O(\varepsilon^2)$$

With a change of variables in the sum over the differences of the future weighting discount function and noting that $\Delta\varepsilon_0 = 0$, we can rewrite this as

$$\ln c_t = \ln c_t^0 + \sum_{i=1}^{t} \frac{\sum_{l=1}^{T-i} D_1^{i+l} \Delta\varepsilon_l}{\sum_{k=i}^{T} D_1^k} + O(\varepsilon^2). \tag{29}$$

Thus the realized utility at time t from the realized consumption path is

$$U_t^* = \sum_{s=t}^{T} D_{s-t} \ln c_s = \sum_{s=t}^{T} D_{s-t} \ln c_s^0 + \sum_{s=t}^{T} \sum_{i=1}^{s} \sum_{l=1}^{T-i} \frac{D_1^{s-t+i+l} \Delta\varepsilon_l}{\sum_{k=i}^{T} D_1^k} + O(\varepsilon^2). \tag{30}$$

Notice that (27) and (30) are the same to zeroth order in the ε_t, so ΔU_t vanishes to zeroth order. Thus we can focus on the first-order terms of (30), which we will call

$$V_t = \sum_{s=t}^{T} \sum_{i=1}^{s} \sum_{l=1}^{T-i} \frac{D_1^{s-t+i+l} \Delta\varepsilon_l}{\sum_{k=i}^{T} D_1^k}.$$

Since V_t is a linear combination of the incremental changes $\Delta\varepsilon_s$ of the future weighting discount function, it is helpful to isolate the effect of each individual change. We define coefficients J_i^t such that

$$V_t = \sum_{i=1}^{T-1} J_i^t \Delta\varepsilon_i. \tag{31}$$

That is to say,

$$J_i^t = \left. \frac{\partial U_t^*}{\partial \Delta\varepsilon_i} \right|_{\varepsilon=0} \tag{32}$$

for $i = 1, \ldots, T-1$, noting that $\Delta\varepsilon_0 = 0$. Thus J_i^t measures how much $\Delta\varepsilon_i$ contributes to the realized utility U_t^* at time t.

In appendix B we derive a convenient expression for the J_i^t:

$$J_i^t = \sum_{s=0}^{T-t} D_1^{i+s} \sum_{j=1}^{\min\{s+t, T-i\}} \frac{1}{\sum_{k=0}^{T-j} D_1^k} > 0, \tag{33}$$

where we have assumed that $D_1 > 0$. Thus an increase in $\Delta\varepsilon_i$ will unambiguously increase U_t^* at $\varepsilon = 0$. Note that if $i = T-1$, since $t \geq 1$, the inner sum in (33) reduces to a single term with $j = 1$, so

$$J_{T-1}^t = D_1^{T-1} \frac{\sum_{s=0}^{T-t} D_1^s}{\sum_{k=0}^{T-1} D_1^k} \leq D_1^{T-1} \leq D_1^{T-t}.$$

Note that the first inequality is strict for $t > 1$.

Another useful property of the J_i^t, shown in appendix C, is that, if $D_1 \in (0,1)$ they are strictly decreasing in i. For $t = 1, \ldots, T$ and $i = 1, \ldots, T-2$,

$$J_i^t > J_{i+1}^t.$$

Intuitively, it would make sense that the contribution of an incremental change $\Delta\varepsilon_i$ to realized utility should get smaller the farther into the future the change in delays from i to $i+1$ gets.

If we express V_t in terms of the ε_i, we can write $\Delta U_t = U_t^* - U_t^c$ as

$$\Delta U_t = J_{T-1}^t \varepsilon_T + \sum_{i=2}^{T-1} (J_{i-1}^t - J_i^t)\varepsilon_i - \sum_{s=t}^{T} D_1^{s-t}\varepsilon_s + O(\varepsilon^2).$$

This shows that, to first order, the difference between the realized utility and the commitment utility at each t is a linear combination of the ε_i for $i = 2, \ldots, T$.

4.1 Comparing the commitment path with the realized path at $t = 1$

The reason why we call the property that we are deriving conditions for in this section "almost Pareto dominance" by the commitment path is an interesting quirk of this discrete-time model. To first-order in the future weighting factors, utility at $t = 1$ is always the same along the realized path and commitment paths. This is a consequence of the fact that at $t = 0$ lifetime utility along the commitment path must, by definition, dominate lifetime utility along any other path, including the realized path, so

$$\Delta U_0 \leq 0.$$

It follows from this that we must have

$$\Delta U_0 = O(\varepsilon^2)$$

since if

$$\Delta U_0 = \sum_{i=2}^{T} \Delta v_i^0 \varepsilon_i + O(\varepsilon^2)$$

for some $v_2^0, \ldots, v_T^0$, not all zero, then there would have to be some choice of the ε_i such that $\Delta U_0 > 0$. Let us define $c_{t,i}^1$ and $c_{t|0,i}^1$ for $i = 2, \ldots, T$ such that

$$c_t = c_t^0 + \sum_{i=2}^{T} c_{t,i}^1 \varepsilon_i + O(\epsilon^2)$$

and

$$c_{t|0} = c_t^0 + \sum_{i=2}^{T} c_{t|0,i}^1 \varepsilon_i + O(\epsilon^2).$$

But since $c_0 = c_{0|0}$,

$$\begin{aligned}
\Delta U_0 &= \ln c_0 + \sum_{t=1}^{T} D_t \ln c_t - \ln c_{0|0} - \sum_{t=1}^{T} D_t \ln c_{t|0} \\
&= \sum_{t=1}^{T} D_1^t (1+\varepsilon_t) \ln \left(\frac{c_t^0 + \sum_{i=2}^{T} c_{t,i}^1 \varepsilon_i}{c_t^0 + \sum_{j=2}^{T} c_{t|0,j}^1 \varepsilon_i} \right) + O(\varepsilon_t^2) \\
&= \sum_{t=1}^{T} D_1^t (1+\varepsilon_t) \sum_{i=2}^{T} \frac{c_{t,i}^1 - c_{t|0,i}^1}{c_t^0} \varepsilon_i + O(\varepsilon_t^2) \\
&= \sum_{t=1}^{T} D_1^t \sum_{i=2}^{T} \frac{c_{t,i}^1 - c_{t|0,i}^1}{c_t^0} \varepsilon_i + O(\varepsilon_t^2).
\end{aligned}$$

Likewise,

$$\begin{aligned}
\Delta U_1 &= \sum_{t=1}^{T} D_{t-1} \ln c_t - \sum_{t=1}^{T} D_{t-1} \ln c_{t|0} \\
&= \sum_{t=1}^{T} D_1^{t-1} \sum_{i=2}^{T} \frac{c_{t,i}^1 - c_{t|0,i}^1}{c_t^0} \varepsilon_i + O(\varepsilon_t^2).
\end{aligned}$$

Thus,

$$\Delta U_1 = \frac{1}{D_1} \Delta U_0 + O(\varepsilon^2).$$

Proposition 4. *If we define $\Delta U_1 = U_1^* - U_1^c$ then*

$$\Delta U_1 = O(\varepsilon^2).$$

This means U_1^c, the utility on the commitment path at period 1, equals U_1^*, the utility on the realized path at period 1, to first order in ε.

The preceding argument does not extend to the second-order terms of ΔU_0 and ΔU_1. Calculating the difference ΔU_1 to second order is beyond the scope of this paper. Caliendo and Findley (2019) have shown for $T = 2$ that ΔU_1 is always positive, which we now understand is a consequence of the second-order term always being positive. Thus it is never possible to have the commitment path dominate the realized path for all t when $T = 2$. For large enough T, however, the second-order term can become negative. The conditions that we establish below for almost Pareto dominance would yield complete Pareto dominance at such large T by the commitment path.

4.2 Comparing the commitment path with the realized path at $t > 1$

To sign the ΔU_t for $t = 2, \ldots, T$, it is helpful to isolate the contribution of the individual future weighting factors, so we define

$$\Delta U_t = \sum_{i=2}^{T} B_i^t \varepsilon_i + O(\varepsilon^2), \tag{34}$$

where the B_i^t represent the rate at which ΔU_t changes with ε_i when all the $\varepsilon_i = 0$:

$$B_i^t = \left. \frac{\partial \Delta U_t}{\partial \varepsilon_i} \right|_{\varepsilon=0}.$$

In other words, the B matrix is the Jacobian of the ΔU_t with respect to the future weights of the discount function. While this may not be immediately apparent, as the following proposition establishes, the signs of the B_i^t are unambiguous.

Proposition 5. *For $t = 2, \ldots, T$, if $D_1 \in (0, 1)$,*

a.

$$B_T^t = J_{T-1}^t - D_1^{T-t} < 0 \tag{35}$$

b. For $t \leq i < T$,

$$B_i^t = J_{i-1}^t - J_i^t - D_1^{i-t} < 0, \tag{36}$$

c. For $2 \leq i < t$,

$$B_i^t = J_{i-1}^t - J_i^t > 0. \tag{37}$$

The proofs of (35) and (37) follow immediately from the properties of the J_i^t detailed above. The proof of (36) is shown in appendix E.

If we write the B_i^t with t indexing the rows of the B matrix and i indexing its columns, since i and t both run from 2 to T, this will be a square matrix. To summarize Proposition 5, the matrix elements along and above the main diagonal will all be negative while the matrix elements below the main diagonal will be positive. This is true irrespective of the difference in definitions between the matrix elements of type a and type c. If we increase ε_i for $i \geq t$, then ΔU_t will decrease. On the other hand, if $i < t$, ΔU_t will increase.

We can understand this result as follows. From (32), $J_{i-1}^t - J_i^t$ is the contribution of ε_i to the realized utility U_t^* for $i = 2, \ldots, T-1$, which we showed in Appendix C is positive. Likewise, J_{T-1}^t is the contribution of ε_T to U_t^* (since there is no ε_{T+1}), and this is also positive. Meanwhile, (24) shows that the contribution of a positive ε_i to the commitment utility U_t^c is D_1^{i-t} for $i \geq t$, which is also positive, and zero otherwise. This last point is the key to the intuition behind Proposition 5. On the commitment path, the only effect of $\varepsilon_i > 0$ is to increase the consumption allocated to time i, so ε_i only contributes to the commitment utility at t if $i \geq t$. In contrast, on the realized path, ε_i contributes to the realized utility at all t for $t = 2, \ldots, T$, regardless of whether i comes before or after t.

The thrust of Proposition 5 is that, whenever ε_i has a nonzero contribution to the commitment utility at t, the contribution of ε_i to the realized utility at t will always be of the same sign. However, if the first-order contribution of ε_i to the commitment utility is nonzero, it will always dominate the contribution of ε_i to the realized utility.

While ε_i only impacts U_t^c through its effect on $\ln c_{i|0}$ and only if $i \geq t$, the effect of ε_i on the realized utility at t is much more complicated since the effect of ε_i is spread over all the period utilities from $\ln c_2$ to $\ln c_T$. What makes this all the more remarkable is that, for example, ε_T only affects the household's decision-making for $t > 0$ through the choice of k_1 at $t = 0$. Notice that at $t = 0$, both the commitment and realized paths start out the same. The only irreversible decision we actually make at $t = 0$ is the decision of how much of our $t = 0$ wealth to allocate to c_0 and how much to divide between $c_1, ..., c_T$. The ε_t will determine how we do this allocation over $c_1, ..., c_T$. Under commitment, we are committing to an allocation where each c_t is strictly a function of ε_t. But when we get to $t = 1$, we do not have to follow the plan that we had at $t = 0$, and we will not if the ε_t are nonzero. Instead, we decide again how we will allocate our wealth at $t = 1$ between c_1 and the $c_2, ..., c_T$.

And likewise, when we get to $t = 2$ we make a new plan for how much to consume at $t = 2$. Thus ε_{T-1} only affects the household's decision-making for $t > 1$ through the choice of k_2 at $t = 1$, and so on. The future weighting factor with the longest delay that appears in (15) at time t is ε_{T-t}. At later times, only weighting factors for shorter delays continue to appear in the Euler equation, so ε_i falls out of the Euler equations for $t > T - i$. Thereafter, a higher ε_i only impacts future consumptions through the choice to consume less at $T - i$. This leaves a bigger pie remaining for the household to allocate amongst its selves at times later than $T - i$.

We can see how this works in relation to figure 1. For example in figure 1b, where $T = 10$ and there is a positive ε_8, there is a big spike in consumption at $t = 8$ on the commitment path. Thus the commitment utility will be higher for the selves that see this spike at $t \leq 8$. On the realized path, consumption is slightly higher for all $t > T - 8 = 2$, so the realized utility is higher for all t. However, the effect of ε_8 on the realized consumptions and the realized utilities is small relative to the effect of ε_8 on $c_{8|0}$ and the commitment utility for $t \leq 8$. On the realized path after $t = 2$, ε_8 drops out of the calculation and provides no further reason for the household to save extra. Therefore it starts to smooth consumption and spreads the extra saving from $t = 2$ over the remaining 8 periods of life. This behavior is also observable in 1a where ε_2 is positive. In that case, starting from $t = 9$, ε_2 drops out and the household spreads the saving among the consumptions over the remaining periods of life. However, we see that since the saving is effectively divided between two periods, as opposed to seven periods in 1b, the increase in the period consumption level is larger. Note that to first order the superposition principle applies, so the effects of the discounting function in total will be the sum of the effects for each of the individual ε_i.

The sign of the matrix elements of type a in Proposition 5 are of most importance for understanding when the commitment path will almost Pareto dominate the realized path or vice versa, so let us focus on why the B_T^t are always negative for $t = 2, \ldots, T$. For the realized path, ε_T only appears in the initial Euler equation (15) that determines $\frac{c_1}{c_0}$, in which $\Delta\varepsilon_{T-1} = \varepsilon_T - \varepsilon_{T-1}$ is discounted by a factor of D_1^{T-1}. The realized utility at t depends on $\ln c_s$ for $s = t, \ldots, T$, which all include $\ln \frac{c_1}{c_0}$. There are $T - t + 1$ such terms, and they are also multiplied by the unperturbed marginal propensity to consume (MPC). After factoring out the unperturbed discount factor D_1^{T-1} mentioned previously, the denominator of the MPC is a sum of $T - 1$ terms of comparable magnitude to the $T - t + 1$ terms they are dividing, which yields a fraction less than one. Thus the magnitude of $\frac{\partial U_t^*}{\partial \varepsilon_T}$ is determined primarily by the discount factor D_1^{T-1}. Since this is smaller than $\frac{\partial U_t^c}{\partial \varepsilon_T} = D_1^{T-t}$, ε_T contributes to the realized utility at t less than it contributes to the commitment utility at t, and $B_{T-1}^t < 0$.

Thus ε_T is of special significance of all the future weighting factors. An increase in ε_T will generate a big spike in consumption at the end of life along the commitment utility that will add to the commitment utility of all the household's selves. However, along the actual realized path, this increase in ε_T will have a more muted effect. The initial self will reduce its consumption to enable the spike it is anticipating at the end of life, but thereafter all of the selves will take a bite of this extra saving. The concentrated dose of consumption in one period would have a bigger impact than spreading the consumption over all of the future selves. Holding the other future discount weights constant, if ε_T is pushed sufficiently high, all of the ΔU_t can be made negative for $t > 1$. Conversely, if ε_T is made sufficiently negative, all of the ΔU_t can be made positive for $t > 1$.[20]

Consequently, we can express the condition for ΔU_t to be negative (positive) in terms of a lower (upper) bound on ε_T. In formal terms, we can rearrange (34) such that $\Delta U_t < 0$ to first order in the future weighting factors iff

$$\varepsilon_T > -\sum_{i=2}^{T-1} \frac{B_i^t \varepsilon_i}{B_T^t},$$

where the direction of the inequality follows from Proposition 5. Let us define the Pareto coefficients

$$P_i^t = -\frac{B_i^t}{B_T^t}$$

for $t = 2, \ldots, T$ and $i = 2, \ldots, T-1$.

Proposition 6. *To first order in ε, a necessary and sufficient condition for the commitment path to Pareto dominate the realized path for all selves except $t = 1$ is that*

$$\varepsilon_T > \sum_{i=2}^{T-1} P_i^t \varepsilon_i.$$

Conversely, a necessary and sufficient condition for the realized path to Pareto dominate the commitment path for all selves except $t = 0, 1$ is that

$$\varepsilon_T < \sum_{i=2}^{T-1} P_i^t \varepsilon_i.$$

Since P_i^t will have the same sign as B_i^t, it also follows from Proposition 5 that $P_i^t < 0$ iff $t \leq i < T$, and $P_i^t > 0$ iff $2 \leq i < t$ for $t = 2, \ldots, T$. If the future weights are all positive, it

[20] Note that in continuous time we cannot vary the terminal future discount weight independently of the other weights while maintaining assumptions about the smoothness of the weighting function. This is one of the main advantages of working in discrete time.

is trivial that $\Delta U_2 < 0$ since the P_i^2 are all negative. Likewise, if the future weights are all negative, $\Delta U_2 > 0$. On the other hand, the P_i^T will all be positive, so the threshold value of ε_T such that $\Delta U_T < 0$ will be strictly positive if the weights are all positive and strictly negative if the weights are all negative. For t in between 2 and T, the signs of the P_i^t will be mixed.

We should emphasize that, while the conditions for the ΔU_t to be positive or negative only specify a threshold value of ε_T, this does not imply that the other future weighting factors have no impact on the conditions for almost Pareto dominance. The threshold values of ε_T, i.e.

$$E_T^t = \sum_{i=2}^{T-1} P_i^t \varepsilon_i$$

for $t = 2, \ldots, T$, are themselves linear functions of the other future weighting factors. While we need to make additional assumptions to guarantee that E_T^T is the tightest threshold, i.e. either the most positive or the most negative, we can see that this threshold must increase (decrease) with all the other future weighting factors if the weights are all positive (negative).

To sum up, the conditions we derived for ΔU_t to be negative for $t = 2, \ldots, T$ together combine to establish a sufficient condition for the commitment path to almost Pareto dominate the realized path. Here we use the term *almost* Pareto dominate to emphasise that the sign of ΔU_1 is determined by second and higher order effects of the ε_t.

Example 7. *In this example we verify our results for a four-period model with T=3. As the first step, we set up the utility of the commitment plan U_t^c, the realized plan U_t^*, and the difference between these two utilities ΔU_t. Then we establish the sufficient condition on ε_3 for the commitment path to almost Pareto dominate the realized path and compare this to the conditions for a concave (convex) log consumption profile.*

In this four-period model,

$$U_t^* = \sum_{s=t}^{3} D_{s-t} \ln(c_s)$$

is the realized utility as of time t in the realized path and

$$U_t^c = \sum_{s=t}^{3} D_{s-t} \ln(c_{s|0})$$

is the utility as of time t if the household commits to its original path. Finally,

$$\Delta U_t = U_t^* - U_t^c = \sum_{s=t}^{3} D_{s-t} \ln\left(\frac{c_s}{c_{s|0}}\right)$$

is the difference between these utilities.

Using (26) and (29) we have

$$\begin{aligned}
\ln\left(\frac{c_1}{c_{1|0}}\right) &= \frac{D_1(1-D_1)\varepsilon_2 + D_1^2\varepsilon_3}{1+D_1+D_1^2} + O(\varepsilon^2) \\
\ln\left(\frac{c_2}{c_{2|0}}\right) &= \frac{D_1(1-D_1)\varepsilon_2 + D_1^2\varepsilon_3}{1+D_1+D_1^2} + \frac{D_1}{1+D_1}\varepsilon_2 - \varepsilon_2 + O(\varepsilon^2) \\
\ln\left(\frac{c_3}{c_{3|0}}\right) &= \frac{D_1(1-D_1)\varepsilon_2 + D_1^2\varepsilon_3}{1+D_1+D_1^2} + \frac{D_1}{1+D_1}\varepsilon_2 - \varepsilon_3 + O(\varepsilon^2)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\Delta U_1 &= \frac{D_1(1-D_1)\varepsilon_2 + D_1^2\varepsilon_3}{1+D_1+D_1^2} + D_1\left[\frac{D_1(1-D_1)\varepsilon_2 + D_1^2\varepsilon_3}{1+D_1+D_1^2} + \frac{D_1}{1+D_1}\varepsilon_2 - \varepsilon_2\right] \\
&\quad + D_1^2\left[\frac{D_1(1-D_1)\varepsilon_2 + D_1^2\varepsilon_3}{1+D_1+D_1^2} + \frac{D_1}{1+D_1}\varepsilon_2 - \varepsilon_3\right] \\
&= D_1(1-D_1)\varepsilon_2 + D_1^2\varepsilon_3 + D_1^2\varepsilon_2 - D_1\varepsilon_2 - D_1^2\varepsilon_3 \\
&= O(\varepsilon^2).
\end{aligned}$$

Since ΔU_1 vanishes to first order,

$$\begin{aligned}
\Delta U_2 &= -\frac{1}{D_1}\ln\left(\frac{c_1}{c_{1|0}}\right) \\
&= -\frac{1}{1+D_1+D_1^2}\left[(1-D_1)\varepsilon_2 + D_1\varepsilon_3\right] + O(\varepsilon^2) \\
\Delta U_3 &= D_1\frac{1-D_1^2+1+D_1+D_1^2}{(1+D_1+D_1^2)(1+D_1)}\varepsilon_2 - \frac{1+D_1}{1+D_1+D_1^2}\varepsilon_3 \\
&= \frac{1+D_1}{1+D_1+D_1^2}\left[\frac{2D_1+D_1^2}{(1+D_1)^2}\varepsilon_2 - \varepsilon_3\right] + O(\varepsilon^2)
\end{aligned}$$

We will have

$$\Delta U_2 < 0$$

to first order of ε if and only if

$$(1 - D_1)\varepsilon_2 + D_1\varepsilon_3 > 0$$

$$\varepsilon_3 > -\frac{1 - D_1}{D_1}\varepsilon_2.$$

Likewise, we will have $\Delta U_3 < 0$ to first order ε if and only if

$$\varepsilon_3 > \left[1 - \frac{1}{(1 + D_1)^2}\right]\varepsilon_2$$

Thus to have $\Delta U_2 < 0$ and $\Delta U_3 < 0$, we need

$$\varepsilon_3 > \max\left\{\left[1 - \frac{1}{(1 + D_1)^2}\right]\varepsilon_2, -\frac{1 - D_1}{D_1}\varepsilon_2\right\} > 0$$

since either both lower bounds are zero (if $\varepsilon_2 = 0$) or one is positive and one is negative. Notice also that

$$P_2^2 = 1 - \frac{1}{(1 + D_1)^2} > 0,$$

and

$$P_2^3 = -\frac{1 - D_1}{D_1} < 0,$$

consistent with Proposition 5 since P_i^t has the same sign as B_i^t.

For comparison, the concavity bounds on ε_2 and ε_3 implied by (22) are $\varepsilon_2 = \Delta\varepsilon_1 > 0$, and

$$\Delta\varepsilon_2 > \frac{D_1}{1 + D_1}\Delta\varepsilon_1 = \frac{D_1}{1 + D_1}\varepsilon_2.$$

Together, these strict concavity bounds on ε_2 and ε_3 imply that

$$\varepsilon_3 > \left[1 + \frac{1}{1 + D_1}\right]\varepsilon_2 > \left[1 - \frac{1}{(1 + D_1)^2}\right]\varepsilon_2 > -\frac{1 - D_1}{D_1}\varepsilon_2.$$

Thus, when $T = 3$, the necessary and sufficient conditions for a concave log consumption profile are also sufficient conditions for the commitment path to almost Pareto dominate the realized path. Likewise, the conditions for a convex log consumption profile are sufficient conditions for the realized path to almost Pareto dominate the commitment path.

5 Discussion

In section 3, we developed the necessary condition for the consumption profile to be concave. And in section 4, we obtained the condition under which the commitment path almost Pareto dominates the realized path. In Example 7, we also saw that the concavity condition for the lifecycle profile of log consumption implies the almost Pareto dominancy of the commitment path over the realized path in a four-period version of the model.

To explore this, we will next show this last result also holds true in a five-period model. Feigenbaum and Raei (2021) established more generally in continuous time that a concave log-consumption profile is a sufficient condition for Pareto dominancy of the commitment path over the realized path. Based on that result and the examples here our conjecture is that this result can be expanded to discrete-time models with a longer horizon ($T > 4$). However, as this next example demonstrates, the complexity of the proofs increases quickly with T, so, if the conjecture is true, a general proof is beyond the scope of the present paper.

A five-period model, If $T = 4$, based on our calculations in section 4, we can obtain the following Pareto bounds for period 2, period 3 and period 4.
The $t = 2$ Pareto bound is

$$\varepsilon_4 > -\frac{1 - D_1}{D_1^2}(\varepsilon_2 + D_1\varepsilon_3).$$

The $t = 3$ Pareto bound is

$$\varepsilon_4 > \frac{3D_1 + D_1^2 + 2D_1^3}{D_1 + D_1^2 + D_1^3}\varepsilon_2 - \frac{1 + D_1}{D_1 + D_1^2 + D_1^3}\varepsilon_3. \tag{38}$$

And the $t = 4$ Pareto bound is

$$\varepsilon_4 > P_2^4\varepsilon_2 + P_3^4\varepsilon_3 = \frac{3D_1 + D_1^2 + 2D_1^3}{(1 + D_1 + D_1^2)^2}\varepsilon_2 + \frac{2D_1^2 + D_1^3 + D_1^4}{(1 + D_1 + D_1^2)^2}\varepsilon_3 \tag{39}$$

Meanwhile, the concavity bounds for $T = 4$ are

$$\varepsilon_4 > \frac{D_1 - D_1^2}{1 + D_1 + D_1^2}\varepsilon_2 + \frac{1 + D_1 + 2D_1^2}{1 + D_1 + D_1^2}\varepsilon_3, \tag{40}$$

$$\varepsilon_3 > \frac{1 + 2D_1}{1 + D_1}\varepsilon_2, \tag{41}$$

and $\varepsilon_2 > 0$.

Let us suppose these concavity bounds are satisfied. Then (40) can be rewritten

$$\begin{aligned}\varepsilon_4 &> \frac{D_1 - D_1^2}{1 + D_1 + D_1^2}\varepsilon_2 + \left(\frac{1 + D_1 + 2D_1^2}{1 + D_1 + D_1^2} - P_3^4\right)\varepsilon_3 + P_3^4\varepsilon_3 \\ &> \frac{D_1 - D_1^2}{1 + D_1 + D_1^2}\varepsilon_2 + \left(\frac{1 + D_1 + 2D_1^2}{1 + D_1 + D_1^2} - P_3^4\right)\frac{1 + 2D_1}{1 + D_1}\varepsilon_2 + P_3^4\varepsilon_3,\end{aligned}$$

where we use (41) to obtain the second inequality. As we show in appendix F,

$$\frac{D_1 - D_1^2}{1 + D_1 + D_1^2} + \left(\frac{1 + D_1 + 2D_1^2}{1 + D_1 + D_1^2} - P_3^4\right)\frac{1 + 2D_1}{1 + D_1} > P_2^4$$

Thus if the concavity bounds are satisfied we have

$$\varepsilon_4 > P_2^4\varepsilon_2 + P_3^4\varepsilon_3,$$

so the $t = 4$ Pareto bound is also satisfied.

Likewise, we can write the Pareto bound at $t = 3$ as

$$\varepsilon_4 > P_2^3\varepsilon_2 + P_3^3\varepsilon_3$$

Suppose that both the concavity bounds and the $t = 4$ Pareto bound (39) are satisfied. We can rewrite the latter as

$$\varepsilon_4 > P_2^4\varepsilon_2 + (P_3^4 - P_3^3)\varepsilon_3 + P_3^3\varepsilon_3$$

Combining this with the concavity bound for ε_3, we obtain

$$\varepsilon_4 > P_2^4\varepsilon_2 + (P_3^4 - P_3^3)\frac{1 + 2D_1}{1 + D_1}\varepsilon_2 + P_3^3\varepsilon_3.$$

We show in appendix G that

$$P_2^4 + (P_3^4 - P_3^3)\frac{1 + 2D_1}{1 + D_1} > P_2^3,$$

so (38) immediately follows.

Thus if the concavity bounds for $T = 4$ are satisfied, the Pareto bounds for $T = 4$ are satisfied.

6 Concluding remarks

Present and future bias are defined as a form of time-inconsistency in which individuals' behavior regarding trade-offs in consumption at the beginning and end of the same time interval vary between the near future and the far future. The common approach for modeling this bias is with a relative discounting function, i.e. a form of discounting function which is a function of the time to consumption from the decision-making present. As a consequence, the optimal plan changes as an individual advances through the life span. A functional form that is widely used in the literature as a proxy for non-exponential discounting functions is the quasihyberbolic functional form, which is used to discuss the shape of the consumption profile and the preferences of different selves.

In this paper we proposed a general representation of relative discounting functions that allows us to focus on how the discounting function deviates from an exponential discounting function that will not exhibit time-inconsistency. We term the perturbation away from the exponential case a *future weighting factor* ε_t. This specific format of the discounting function provides a simple way to depict a future bias by having all ε_t be negative and decreasing for $t > 1$, and a present bias by having all ε_t be positive and increasing for $t > 1$. We find that the former is a necessary condition to have a convex log consumption profile and the latter is a necessary condition to have a concave log consumption profile.

Also, using the proposed future weighting functional form, we find a condition on ε_t under which the consumption profile that is determined in the first period of life will Pareto dominate the consumption profiles that are chosen at each period, starting from period two. This result is especially useful because this Pareto dominance is often used to motivate how one performs welfare analysis in these models with time-inconsistent preferences, where choosing a reference consumption plan for the analysis is a point of controversy in the literature.

An interesting extension of this paper would be to further explore the relation between the condition for the concavity of the log-consumption and the condition for the Pareto dominance of the commitment path. Providing a simple proof that can be extended beyond a five-period model–or, if the result is not true, finding a counterexample–would be very helpful.

References

Orazio Attanasio and Martin Browning. Consumption over the life cycle and over the business cycle, 1993.

Orazio Attanasio and Guglielmo Weber. Is consumption growth consistent with intertemporal optimization? evidence from the consumer expenditure survey. Journal of Political Economy, 103(6):1121–57, 1995. URL https://EconPapers.repec.org/RePEc:ucp:jpolec:v:103:y:1995:i:6:p:1121-57.

Orazio P. Attanasio, James Banks, Costas Meghir, and Guglielmo Weber. Humps and bumps in lifetime consumption. Journal of Business Economic Statistics, 17(1):22–35, 1999. ISSN 07350015. URL http://www.jstor.org/stable/1392236.

B. Douglas Bernheim and Debraj Ray. Economic Growth with Intergenerational Altruism. The Review of Economic Studies, 54(2):227–243, 04 1987. ISSN 0034-6527. doi: 10.2307/2297513. URL https://doi.org/10.2307/2297513.

Martin Browning and Thomas Crossley. Unemployment insurance benefit levels and consumption changes. Journal of Public Economics, 80(1):1–23, 2001. URL https://EconPapers.repec.org/RePEc:eee:pubeco:v:80:y:2001:i:1:p:1-23.

Martin Browning, Angus Deaton, and Margaret Irish. A profitable approach to labor supply and commodity demands over the life-cycle. Econometrica: journal of the econometric society, pages 503–543, 1985.

James Bullard and James Feigenbaum. A leisurely reading of the life-cycle consumption data. Journal of Monetary Economics, 54(8):2305–2320, 2007. URL https://EconPapers.repec.org/RePEc:eee:moneco:v:54:y:2007:i:8:p:2305-2320.

Frank Caliendo and David Aadland. Short-term planning and the life-cycle consumption puzzle. Journal of Economic Dynamics and Control, 31(4):1392–1415, 2007. ISSN 0165-1889. doi: https://doi.org/10.1016/j.jedc.2006.05.002. URL https://www.sciencedirect.com/science/article/pii/S016518890600100X.

Frank Caliendo and T. Scott Findley. Commitment and welfare. Journal of Economic Behavior and Organization, 2019.

John Y Campbell and N Gregory Mankiw. Consumption, income, and interest rates: Reinterpreting the time series evidence. NBER macroeconomics annual, 4:185–216, 1989.

Dan Cao and Iván Werning. Saving and dissaving with hyperbolic discounting. Econometrica, 86(3):805–857, 2018. doi: https://doi.org/10.3982/ECTA15112. URL https://onlinelibrary.wiley.com/doi/abs/10.3982/ECTA15112.

Christopher Carroll. Buffer-stock saving and the life cycle/permanent income hypothesis. The Quarterly Journal of Economics, 112(1):1–55, 1997. URL https://EconPapers.repec.org/RePEc:oup:qjecon:v:112:y:1997:i:1:p:1-55.

Christopher Carroll and Lawrence Summers. Consumption growth parallels income growth: Some new evidence. In National Saving and Economic Performance, pages 305–348. National Bureau of Economic Research, Inc, 1991. URL https://EconPapers.repec.org/RePEc:nbr:nberch:5995.

Christopher D Carroll. How does future income affect current consumption? The Quarterly Journal of Economics, 109(1):111–147, 1994.

Angus Deaton. Understanding Consumption. Oxford University Press, 1992. URL https://EconPapers.repec.org/RePEc:oxp:obooks:9780198288244.

Mathias Dewatripont, Isabelle Brocas, and Juan Carrillo. Commitment devices under self-control problems: an overview. Ulb institutional repository, ULB – Universite Libre de Bruxelles, 2004.

Nicolas Drouhin. Non-stationary additive utility and time consistency. Journal of Mathematical Economics, 86:1 – 14, 2020. ISSN 0304-4068. doi: https://doi.org/10.1016/j.jmateco.2019.10.005. URL http://www.sciencedirect.com/science/article/pii/S0304406819301077.

Keith Marzilli Ericson and David Laibson. Chapter 1 - intertemporal choice. In B. Douglas Bernheim, Stefano DellaVigna, and David Laibson, editors, Handbook of Behavioral Economics - Foundations and Applications 2, volume 2 of Handbook of Behavioral Economics: Applications and Foundations 1, pages 1–67. North-Holland, 2019. doi: https://doi.org/10.1016/bs.hesbe.2018.12.001. URL https://www.sciencedirect.com/science/article/pii/S2352239918300253.

James Feigenbaum. Can mortality risk explain the consumption hump? Journal of Macroeconomics, 30(3):844–872, 2008.

James Feigenbaum and Sepideh Raei. Deviation from exponential discounting and present bias in continuous time. working paper, 2021. URL http://www.platonicadventures.com/research.html.

Martin Feldstein. The optimal level of social security benefits. The Quarterly Journal of Economics, 100(2):303–320, 1985.

Jesus Fernandez-Villaverde and Dirk Krueger. Consumption and saving over the life cycle: How important are consumer durables? Macroeconomic Dynamics, 15(5):725–770, 2011. URL https://EconPapers.repec.org/RePEc:cup:macdyn:v:15:y:2011:i:05:p:725-770_00.

Milton Friedman. Theory of the consumption function. Princeton university press, 2018.

Pierre-Olivier Gourinchas and Jonathan A. Parker. Consumption over the life cycle. Econometrica, 70(1):47–89, 2002. doi: https://doi.org/10.1111/1468-0262.00269. URL https://onlinelibrary.wiley.com/doi/abs/10.1111/1468-0262.00269.

Steven R Grenadier and Neng Wang. Investment under uncertainty and time-inconsistent preferences. Journal of Financial Economics, 84(1):2–39, 2007.

Barbara Griffin, Beryl Hesketh, and Vanessa Loh. The influence of subjective life expectancy on retirement transition and planning: a longitudinal study. Journal of Vocational Behavior, 81(2):129–137, 2012.

Gary Hansen and Selahattin Imrohoroglu. Consumption over the Life Cycle: The Role of Annuities. Review of Economic Dynamics, 11(3):566–583, July 2008. doi: 10.1016/j.red.2007.12.004. URL https://ideas.repec.org/a/red/issued/06-155.html.

Christopher Harris and David Laibson. Instantaneous gratification. The Quarterly Journal of Economics, 128(1):205–248, 2013.

James Heckman. Life cycle consumption and labor supply: An explanation of the relationship between income and consumption over the life cycle. American Economic Review, 64(1):188–94, 1974. URL https://EconPapers.repec.org/RePEc:aea:aecrev:v:64:y:1974:i:1:p:188-94.

Eunice Hong and Sherman D Hanna. Financial planning horizon: A measure of time preference or a situational factor? Journal of Financial Counseling and Planning, 25(2):184–196, 2014.

R Glenn Hubbard, Jonathan Skinner, and Stephen P Zeldes. The importance of precautionary motives in explaining individual and aggregate saving. In Carnegie-Rochester conference series on public policy, volume 40, pages 59–125. Elsevier, 1994.

David Laibson. Hyperbolic discounting and consumption. PhD diss. Massachusetts Institute of Technology, 1994.

David Laibson. Golden eggs and hyperbolic discounting. The Quarterly Journal of Economics, 112(2):443–478, 1997.

David Laibson. Life-cycle consumption and hyperbolic discount functions. European Economic Review, 42(3):861 – 871, 1998.

David I Laibson. Hyperbolic discount functions, undersaving, and savings policy. Working Paper 5635, National Bureau of Economic Research, June 1996.

David I. Laibson, Andrea Repetto, Jeremy Tobacman, Robert E. Hall, William G. Gale, and George A. Akerlof. Self-control and saving for retirement. Brookings Papers on Economic Activity, 1998(1):91–196, 1998.

John Lane and Tapan Mitra. On nash equilibrium programs of capital accumulation under altruistic preferences. International Economic Review, 22(2):309–331, 1981. ISSN 00206598, 14682354. URL `http://www.jstor.org/stable/2526279`.

Wolfgang Leininger. The Existence of Perfect Equilibria in a Model of Growth with Altruism between Generations. Review of Economic Studies, 53(3):349–367, 1986. URL `https://ideas.repec.org/a/oup/restud/v53y1986i3p349-367..html`.

Jesus Marin-Solano and Jorge Navas. Non-constant discounting in finite horizon: The free terminal time case. Journal of Economic Dynamics and Control, 33(3):666–675, 2009.

Franco Modigliani and Richard Brumberg. Utility analysis and the consumption function: An interpretation of cross-section data. Franco Modigliani, 1(1):388–436, 1954.

Congming Mu, Jinqiang Yang, et al. Optimal contract theory with time-inconsistent preferences. Economic Modelling, 52:519–530, 2016.

Keizo Nagatani. Life cycle saving: theory and fact. The American Economic Review, 62(3): 344–353, 1972.

Ted O'Donoghue and Matthew Rabin. The economics of immediate gratification. Journal of Behavioral Decision Making, 13(2):233–250.

Ted O'Donoghue and Matthew Rabin. Doing it now or later. American Economic Review, 89(1):103–124, March 1999.

Ted O'Donoghue and Matthew Rabin. Choice and procrastination. The Quarterly Journal of Economics, 116(1):121–160, 2001.

Ted O'Donoghue and Matthew Rabin. Present bias: Lessons learned and to be learned. American Economic Review, 105(5):273–79, May 2015.

Edmund S Phelps and Robert A Pollak. On second-best national saving and game-equilibrium growth. The Review of Economic Studies, 35(2):185–199, 1968.

Drazen Prelec. Decreasing impatience: A criterion for non-stationary time preference and "hyperbolic" discounting. The Scandinavian Journal of Economics, 106(3):511–532, 2004. doi: https://doi.org/10.1111/j.0347-0520.2004.00375.x. URL `https://onlinelibrary.wiley.com/doi/abs/10.1111/j.0347-0520.2004.00375.x`.

Michael Richter. A time-inconsistent first welfare theorem:efficiency and the convexity of patience. working paper, 2020.

Paul A. Samuelson. A note on measurement of utility. The Review of Economic Studies, 4 (2):155–161, 1937.

R. H. Strotz. Myopia and inconsistency in dynamic utility maximization. Review of Economic Studies, 23(3):165–180, 1955a.

Robert Henry Strotz. Myopia and inconsistency in dynamic utility maximization. The review of economic studies, 23(3):165–180, 1955b.

Lester C. Thurow. The optimum lifetime distribution of consumption expenditures. The American Economic Review, 59(3):324–330, 1969. ISSN 00028282. URL `http://www.jstor.org/stable/1808961`.

Appendices

A Simplifying the Concavity Condition

The log consumption profile is concave at $t+1$ iff we have

$$\frac{\sum_{z'=1}^{T-t-1} D_1^{z'}(1+\varepsilon_{z'})}{\sum_{s'=1}^{T-t} D_1^{s'}(1+\varepsilon_{s'})} \frac{\sum_{s=1}^{T-t} D_1^{s}(1+\varepsilon_{s-1})}{\sum_{z=1}^{T-t-1} D_1^{z}(1+\varepsilon_{z-1})} \leq 1.$$

We can rearrange this inequality as follows.

$$\frac{\sum_{z'=1}^{T-t-1} D_1^{z'}(1+\varepsilon_{z'})}{\sum_{s'=1}^{T-t} D_1^{s'}(1+\varepsilon_{s'})} \leq \frac{\sum_{z=1}^{T-t-1} D_1^{z}(1+\varepsilon_{z-1})}{\sum_{s=1}^{T-t} D_1^{s}(1+\varepsilon_{s-1})}$$

$$1-\frac{D_1^{T-t}(1+\varepsilon_{T-t})}{\sum_{s'=1}^{T-t} D_1^{s'}(1+\varepsilon_{s'})} \leq 1-\frac{D_1^{T-t}(1+\varepsilon_{T-t-1})}{\sum_{s=1}^{T-t} D_1^{s}(1+\varepsilon_{s-1})}$$

$$\frac{1+\varepsilon_{T-t-1}}{\sum_{s=1}^{T-t} D_1^{s}(1+\varepsilon_{s-1})} \leq \frac{1+\varepsilon_{T-t}}{\sum_{s'=1}^{T-t} D_1^{s'}(1+\varepsilon_{s'})}$$

We wish to isolate ε_{T-t}, which appears in both the numerator and the denominator of the right-hand side.

$$\frac{1+\varepsilon_{T-t-1}}{\sum_{s=0}^{T-t-1} D_1^{s+1}(1+\varepsilon_{s})} \leq \frac{1+\varepsilon_{T-t}}{\sum_{s'=1}^{T-t-1} D_1^{s'}(1+\varepsilon_{s'})+D_1^{T-t}(1+\varepsilon_{T-t})}$$

$$\frac{\sum_{s'=1}^{T-t-1} D_1^{s'}(1+\varepsilon_{s'})+D_1^{T-t}(1+\varepsilon_{T-t})}{\sum_{s=0}^{T-t-1} D_1^{s+1}(1+\varepsilon_{s})}(1+\varepsilon_{T-t-1}) \leq 1+\varepsilon_{T-t}$$

$$\frac{\sum_{s'=1}^{T-t-1} D_1^{s'}(1+\varepsilon_{s'})}{\sum_{s=0}^{T-t-1} D_1^{s+1}(1+\varepsilon_{s})}(1+\varepsilon_{T-t-1}) \leq (1+\varepsilon_{T-t})\left[1-\frac{D_1^{T-t}(1+\varepsilon_{T-t-1})}{\sum_{s=0}^{T-t-1} D_1^{s+1}(1+\varepsilon_{s})}\right]$$

$$\frac{\sum_{s'=1}^{T-t-1} D_1^{s'}(1+\varepsilon_{s'})}{\sum_{s=0}^{T-t-1} D_1^{s+1}(1+\varepsilon_{s})}(1+\varepsilon_{T-t-1}) \leq (1+\varepsilon_{T-t})\left[\frac{\sum_{z'=0}^{T-t-2} D_1^{z'+1}(1+\varepsilon_{z'})}{\sum_{z=0}^{T-t-1} D_1^{z+1}(1+\varepsilon_{z})}\right]$$

Thus we obtain the condition

$$\frac{\sum_{z=1}^{T-t-1} D_1^{z}(1+\varepsilon_{z})}{\sum_{z'=0}^{T-t-2} D_1^{z'+1}(1+\varepsilon_{z'})}(1+\varepsilon_{T-t-1}) \leq 1+\varepsilon_{T-t} \tag{42}$$

for concavity at $t+1$.

B Different Expressions for J_i^t notation

we know that

$$\Delta U_t = \sum_{s=t}^{T} \sum_{i=0}^{s-1} \sum_{j=1}^{T-i-1} \frac{D_1^{j+s-t} \Delta\varepsilon_j}{\sum_{k=0}^{T-i-1} D_1^k} - \sum_{s=t}^{T} D_1^{s-t} \varepsilon_s + O(\varepsilon^2)$$

Let

$$V_t^T = \sum_{s=t}^{T} \sum_{i=0}^{s-1} \sum_{j=1}^{T-i-1} \frac{D_1^{j+s-t} \Delta\varepsilon_j}{\sum_{k=0}^{T-i-1} D_1^k}$$

$$V_t^T = \sum_{s=0}^{T-t} \sum_{i=0}^{s+t-1} \sum_{j=1}^{T-i-1} \frac{D_1^{j+s} \Delta\varepsilon_j}{\sum_{k=0}^{T-i-1} D_1^k}$$

If we switch the first two summations, the s summation will commute with the j summation. Note that $t \geq 1$.

Let $S = \{(s,i) : 0 \leq s \leq T-t \wedge 0 \leq i \leq s+t-1\}$, so i runs from 0 to $T-1$. Let $S' = \{(s,i) : 0 \leq i \leq T-1 \wedge \max\{i+1-t, 0\} \leq s \leq T-t\}$. Let $(s,i) \in S$. Then $0 \leq s \leq T-t \wedge 0 \leq i \leq s+t-1 \leq T-1\}$. And we have $i+1-t \leq s \leq T-t$. We also have $0 \leq s \leq T-t$, so $\max\{0, i+1-t\} \leq s \leq T-t$. Thus $(s,i) \in S'$. Let $(s,i) \in S'$. Then $0 \leq i \leq T-1 \wedge \max\{i+1-t, 0\} \leq s \leq T-t$. Then $0 \leq s \leq T-t$. We also have $0 \leq i$ and $i+1-t \leq s$ so $i \leq s+t-1$. Therefore $(s,i) \in S$. Thus

$$V_t^T = \sum_{i=0}^{T-1} \sum_{s=\max\{0,i+1-t\}}^{T-t} \sum_{j=1}^{T-i-1} \frac{D_1^{j+s} \Delta\varepsilon_j}{\sum_{k=0}^{T-i-1} D_1^k}$$

$$V_t^T = \sum_{i=0}^{T-1} \sum_{j=1}^{T-i-1} D_1^j \Delta\varepsilon_j \frac{\sum_{s=\max\{0,i+1-t\}}^{T-t} D_1^s}{\sum_{k=0}^{T-i-1} D_1^k}$$

When $i = T-1$, the inner sum vanishes, so

$$V_t^T = \sum_{i=0}^{T-2} \sum_{j=1}^{T-i-1} D_1^j \Delta\varepsilon_j \frac{\sum_{s=\max\{0,i+1-t\}}^{T-t} D_1^s}{\sum_{k=0}^{T-i-1} D_1^k}$$

Let $i' = i+1$, so $i = i'-1$

$$V_t^T = \sum_{i'=1}^{T-1} \sum_{j=1}^{T-i'} D_1^j \Delta\varepsilon_j \frac{\sum_{s=\max\{0,i\prime-t\}}^{T-t} D_1^s}{\sum_{k=0}^{T-i'} D_1^k}$$

$$V_t^T = \sum_{i=1}^{T-1} \sum_{j=1}^{T-i} D_1^j \Delta\varepsilon_j \frac{\sum_{s=\max\{0,i-t\}}^{T-t} D_1^s}{\sum_{k=0}^{T-i} D_1^k}$$

$$V_t^T = \sum_{j=1}^{T-1} \sum_{i=1}^{T-j} D_1^j \Delta\varepsilon_j \frac{\sum_{s=\max\{0,i-t\}}^{T-t} D_1^s}{\sum_{k=0}^{T-i} D_1^k}$$

Switching the roles of i and j, we get

$$V_t^T = \sum_{i=1}^{T-1} \sum_{j=1}^{T-i} D_1^i \Delta\varepsilon_i \frac{\sum_{s=\max\{0,j-t\}}^{T-t} D_1^s}{\sum_{k=0}^{T-j} D_1^k}$$

Let us define

$$J_i^t = \sum_{j=1}^{T-i} \frac{\sum_{s=\max\{0,j-t\}}^{T-t} D_1^{i+s}}{\sum_{k=0}^{T-j} D_1^k} \tag{43}$$

Then

$$V_t^T = \sum_{i=1}^{T-1} J_i^t \Delta\varepsilon_i \tag{44}$$

Thus

$$\begin{aligned} V_t^T &= \sum_{i=1}^{T-1} J_i^t(\varepsilon_{i+1} - \varepsilon_i) \\ &= \sum_{i=1}^{T-1} J_i^t \varepsilon_{i+1} - \sum_{i=1}^{T-1} J_i^t \varepsilon_i \\ &= \sum_{i=2}^{T} J_{i-1}^t \varepsilon_i - \sum_{i=2}^{T-1} J_i^t \varepsilon_i \\ &= J_{T-1}^t \varepsilon_T + \sum_{i=2}^{T-1} (J_{i-1}^t - J_i^t)\varepsilon_i \end{aligned}$$

$$\Delta U_t = V_t^T - \sum_{s=t}^{T} D_1^{s-t} \varepsilon_s + O(\varepsilon^2)$$

$$\Delta U_t = J_{T-1}^t \varepsilon_T + \sum_{i=2}^{T-1} (J_{i-1}^t - J_i^t)\varepsilon_i - \sum_{s=t}^{T} D_1^{s-t} \varepsilon_s + O(\varepsilon^2) \tag{45}$$

We can show that the way we define J_i^t notation in the paper is equivalent to the formula we have here by switching the indices in the following way. Let us start with the definition of j_i^t that we used above:

$$J_i^t = \sum_{j=1}^{T-i} \frac{\sum_{s=\max\{0,j-t\}}^{T-t} D_1^{i+s}}{\sum_{k=0}^{T-j} D_1^k}$$

Let $S = \{(j,s) : 1 \le j \le T-i \wedge \max\{0, j-t\} \le s \le T-t\}$. Let $S' = \{(j,s) : 0 \le s \le T-t \wedge 1 \le j \le \min\{s+t, T-i\}\}$. Let $(j,s) \in S$. Then $1 \le j \le T-i \wedge \max\{0, j-t\} \le s \le T-t$. Then we have $0 \le s \le T-t$. We also have $1 \le j \le T-i$. And we have $j-t \le s$, so $1 \le j \le \min\{s+t, T-i\}$. Thus $(j,s) \in S'$.

Suppose $(j,s) \in S'$. Then we have $0 \le s \le T-t \wedge 1 \le j \le \min\{s+t, T-i\}$. Thus $1 \le j \le \min\{s+t, T-i\}$. We also have $0 \le s$ and $j \le s+t$, so $s \ge \max\{0, j-t\}$, and so $\max\{0, j-t\} \le s \le T-t$. Thus we can write

$$J_i^t = \sum_{s=0}^{T-t} D_1^{i+s} \sum_{j=1}^{\min\{s+t,T-i\}} \frac{1}{\sum_{k=0}^{T-j} D_1^k}. \tag{46}$$

C J_i^t is strictly decreasing in i

$$J_i^t = \sum_{s=0}^{T-t} D_1^{i+s} \sum_{j=1}^{\min\{s+t,T-i\}} \frac{1}{\sum_{k=0}^{T-j} D_1^k}$$

$$\begin{aligned} J_{i+1}^t &= \sum_{s=0}^{T-t} D_1^{i+1+s} \sum_{j=1}^{\min\{s+t,T-i-1\}} \frac{1}{\sum_{k=0}^{T-j} D_1^k} \\ &\le \sum_{s=0}^{T-t} D_1^{i+1+s} \sum_{j=1}^{\min\{s+t,T-i\}} \frac{1}{\sum_{k=0}^{T-j} D_1^k} \\ &< \sum_{s=0}^{T-t} D_1^{i+s} \sum_{j=1}^{\min\{s+t,T-i\}} \frac{1}{\sum_{k=0}^{T-j} D_1^k} = J_i^t, \end{aligned}$$

where this last inequality assumes $D_1 \in (0,1)$.

D Direct Proof of Proposition 4

The proof amounts to showing that the first-order expansion of equilibrium utility at $t = 1$ is

$$V_1 = \sum_{s=2}^{T} D_1^{s-1} \varepsilon_s, \tag{47}$$

where the right-hand side is lifetime utility at $t = 1$ under commitment, also to first order.

In Appendix B, we show that another expression for J_i^t is

$$J_i^t = \sum_{j=1}^{T-i} \frac{\sum_{s=\max\{0,j-t\}}^{T-t} D_1^{i+s}}{\sum_{k=0}^{T-j} D_1^k}.$$

For $t = 1$, this simplifies to

$$J_i^1 = \sum_{j=1}^{T-i} \frac{\sum_{s=j-1}^{T-1} D_1^{i+s}}{\sum_{k=0}^{T-j} D_1^k}. \tag{48}$$

We can factor D_1^{i+j-1} out of the summation over s, leaving

$$\frac{\sum_{k'=0}^{T-j} D_1^{k'}}{\sum_{k=0}^{T-j} D_1^k} = 1.$$

Thus

$$J_i^1 = D_1^i \sum_{j=1}^{T-i} D_1^{j-1},$$

and from (31)

$$V_1^T = \sum_{i=1}^{T-1} \sum_{j=1}^{T-i} D_1^{i+j-1} \Delta\varepsilon_i.$$

Let $l = i + j$, so $j = l - i$. Then

$$V_1^T = \sum_{i=1}^{T-1} \sum_{l=i+1}^{T} D_1^{l-1} \Delta\varepsilon_i.$$

Finally, if we commute the sums,

$$V_1^T = \sum_{l=2}^{T} D_1^{l-1} \sum_{i=1}^{l-1} \Delta\varepsilon_i,$$

the result follows from the fact that

$$\varepsilon_l = \sum_{i=1}^{l-1} \Delta\varepsilon_i.$$

E Proof of Inequality (36)

For $t = 2, ..., T$ and $i = 1, ..., T-1$, let us define

$$M_i^t = J_{i-1}^t - J_i^t. \tag{49}$$

Using (33),

$$M_i^t = D_1^{i-1}\left[(1-D_1)\sum_{s=0}^{T-t} D_1^s \sum_{j=1}^{\min\{s+t,T-i\}} \frac{1}{\sum_{k=0}^{T-j} D_1^k} + \frac{\sum_{s=\max\{T-i-t+1,0\}}^{T-t} D_1^s}{\sum_{k=0}^{i-1} D_1^k}\right]. \tag{50}$$

An equivalent but more convenient expression is obtained by rearranging the sums in the first term:

$$M_i^t = D_1^{i-1}\left[(1-D_1)\sum_{j=1}^{T-i} \frac{\sum_{s=\max\{0,j-t\}}^{T-t} D_1^s}{\sum_{k=0}^{T-j} D_1^k} + \frac{\sum_{s=\max\{T-i-t+1,0\}}^{T-t} D_1^s}{\sum_{k=0}^{i-1} D_1^k}\right] \tag{51}$$

Let

$$S = \{(s,j) : 0 \leq s \leq T-t \wedge 1 \leq j \leq \min\{s+t, T-i\}\}$$

and

$$S' = \{(s,j) : 1 \leq j \leq T-i \wedge \max\{0, j-t\} \leq s \leq T-t\}.$$

Suppose that $(s,j) \in S$. Then $0 \leq s \leq T-t \wedge 1 \leq j \leq \min\{s+t, T-i\}$. Thus $1 \leq j \leq T-i$ and $0 \leq s \leq T-t$. Plus $j \leq s+t$, so $s \geq j-t$. Thus $\max\{0, j-t\} \leq s \leq T-t$. Therefore, $(s,j) \in S'$.

Suppose that $(s,j) \in S'$. Then $1 \leq j \leq T-i \wedge \max\{0, j-t\} \leq s \leq T-t\}$. Thus $0 \leq s \leq T-t$. And $1 \leq j \leq T-i$, and $j-t \leq s$, so $j \leq s+t$. Therefore, $1 \leq j \leq \min\{s+t, T-i\}$, so $(s,j) \in S$. Thus, we can rewrite (50) as (51).

If $D_1 \in (0,1)$ then we can write

$$M_s^t < D_1^{s-1}\left[(1-D_1)\sum_{j=1}^{T-s}\frac{\sum_{i=j-t}^{T-t} D_1^i}{\sum_{k=0}^{T-j} D_1^k} + \frac{\sum_{i=T-s-t+1}^{T-t} D_1^i}{\sum_{k=0}^{s-1} D_1^k}\right].$$

This inequality is strict because $t \geq 2$, so there will be at least one positive term with $s < 0$ in the first sum that is not included in the first sum of (51). Then

$$\begin{aligned} M_s^t &< D_1^{s-1}\left[(1-D_1)\sum_{j=1}^{T-s} D_1^{j-t}\frac{\sum_{i=0}^{T-j} D_1^i}{\sum_{k=0}^{T-j} D_1^k} + D_1^{T-s-t+1}\frac{\sum_{i=0}^{i-1} D_1^i}{\sum_{k=0}^{s-1} D_1^k}\right] \\ &= D_1^{s-1}\left[(1-D_1)\sum_{j=1}^{T-s} D_1^{j-t} + D_1^{T-s-t+1}\right] \\ &= (1-D_1)D_1^{s-1}\sum_{j=1}^{T-s} D_1^{j-t} + D_1^{T-t} \end{aligned}$$

We can use this result to determine the sign of B_s^t for $t \leq s < T$. Since $s \geq t$,

$$\begin{aligned} B_s^t &= M_s^t - D_1^{s-t} \\ &< (1-D_1)D_1^{s-1}\sum_{j=1}^{T-s} D_1^{j-t} + D_1^{T-t} - D_1^{s-t} \\ &= (1-D_1)\left[D_1^{s-1}\sum_{j=1}^{T-s} D_1^{j-t} + D_1^{s-t}\frac{D_1^{T-t-s+t}-1}{1-D_1}\right] \\ &= (1-D_1)\left[D_1^{s-1}\sum_{j=1}^{T-s} D_1^{j-t} + D_1^{s-t}\frac{D_1^{T-s}-1}{1-D_1}\right] \\ &= (1-D_1)\sum_{j=1}^{T-s}\left[D_1^{s-1}D_1^{j-t} - D_1^{s-t}D_1^{j-1}\right] = 0 \end{aligned}$$

F Proof of inequality (5)

we defined P_2^4 as

$$P_2^4 = \frac{3D_1 + D_1^2 + 2D_1^3}{(1+D_1+D_1^2)^2} \tag{52}$$

therefore

$$\begin{aligned} P_2^4 - \frac{D_1 - D_1^2}{1 + D_1 + D_1^2} &= \frac{3D_1 + D_1^2 + 2D_1^3 - D_1(1 - D_1)(1 + D_1 + D_1^2)}{(1 + D_1 + D_1^2)^2} \\ &= \frac{3D_1 + D_1^2 + 2D_1^3 - D_1(1 - D_1^3)}{(1 + D_1 + D_1^2)^2} \\ &= \frac{3D_1 + D_1^2 + 2D_1^3 - D_1 + D_1^4}{(1 + D_1 + D_1^2)^2} \\ &= \frac{2D_1 + D_1^2 + 2D_1^3 + D_1^4}{(1 + D_1 + D_1^2)^2} \geq 0 \end{aligned}$$

also we have

$$\begin{aligned} (1 + D_1 + D_1^2)^2 &= (1 + D_1)^2 + 2D_1^2(1 + D_1) + D_1^4 \\ &= 1 + 2D_1 + D_1^2 + 2D_1^2 + 2D_1^3 + D_1^4 \\ &= 1 + 2D_1 + 3D_1^2 + 2D_1^3 + D_1^4 \end{aligned}$$

hence

$$P_2^4 - \frac{D_1 - D_1^2}{1 + D_1 + D_1^2} = \frac{2D_1 + D_1^2 + 2D_1^3 + D_1^4}{(1 + D_1 + D_1^2)^2} < \frac{1 + 2D_1 + 3D_1^2 + 2D_1^3 + D_1^4}{(1 + D_1 + D_1^2)^2} = 1 \tag{53}$$

with equality only if $D_1 = 0$.

Also, we defined P_3^4 as

$$P_3^4 = \frac{2D_1^2 + D_1^3 + D_1^4}{(1 + D_1 + D_1^2)^2}$$

therefore we can have the following

$$\begin{aligned}
\left(\frac{1+D_1+2D_1^2}{1+D_1+D_1^2}-P_3^4\right)\frac{1+2D_1}{1+D_1} &= \frac{(1+D_1)(1+D_1+D_1^2+D_1^3)}{(1+D_1+D_1^2)^2}\frac{1+2D_1}{1+D_1} \\
&= \frac{(1+2D_1)(1+D_1+D_1^2+D_1^3)}{(1+D_1+D_1^2)^2} \\
&= \frac{1+D_1+D_1^2+D_1^3+2D_1+2D_1^2+2D_1^3+2D_1^4}{(1+D_1+D_1^2)^2} \\
&= \frac{1+3D_1+3D_1^2+3D_1^3+2D_1^4}{(1+D_1+D_1^2)^2} \\
&= 1+\frac{D_1+D_1^3+D_1^4}{(1+D_1+D_1^2)^2} \\
&> 1 > P_2^4-\frac{D_1-D_1^2}{1+D_1+D_1^2}
\end{aligned}$$

in which we used (53) to drive the last inequality.

G Proof of inequality (5)

Here we show that $h(D_1)$ which is defined as

$$h(D_1)=(P_3^4-P_3^3)\frac{1+2D_1}{1+D_1}+P_2^4$$

satisfies

$$h(D_1)-P_2^3>0$$

As a reminder,

$$P_2^3=\frac{3+D_1+2D_1^2}{1+D_1+D_1^2}>\frac{3D_1+D_1^2+2D_1^3}{(1+D_1+D_1^2)^2}=P_2^4$$

and

$$P_3^3=-\frac{1+D_1}{D_1+D_1^2+D_1^3}<0<\frac{2D_1^2+D_1^3+D_1^4}{(1+D_1+D_1^2)^2}=P_3^4$$

$$
\begin{aligned}
P_3^4 - P_3^3 &= \frac{2D_1^2 + D_1^3 + D_1^4}{(1+D_1+D_1^2)^2} + \frac{1+D_1}{D_1+D_1^2+D_1^3} \\
&= \frac{2D_1^3 + D_1^4 + D_1^5 + (1+D_1)(1+D_1+D_1^2)}{D_1(1+D_1+D_1^2)^2} \\
&= \frac{2D_1^3 + D_1^4 + D_1^5 + 1 + 2D_1 + 2D_1^2 + D_1^3}{D_1(1+D_1+D_1^2)^2} \\
&= \frac{1 + 2D_1 + 2D_1^2 + 3D_1^3 + D_1^4 + D_1^5}{D_1(1+D_1+D_1^2)^2}
\end{aligned}
$$

and

$$
\begin{aligned}
P_2^4 - P_2^3 &= \frac{3D_1 + D_1^2 + 2D_1^3}{(1+D_1+D_1^2)^2} - \frac{3+D_1+2D_1^2}{1+D_1+D_1^2} \\
&= \frac{3+D_1+2D_1^2}{1+D_1+D_1^2}\left[\frac{D_1}{1+D_1+D_1^2} - 1\right] \\
&= -\frac{(3+D_1+2D_1^2)(1+D_1^2)}{(1+D_1+D_1^2)^2} \\
&= -\frac{3+D_1+2D_1^2+3D_1^2+D_1^3+2D_1^4}{(1+D_1+D_1^2)^2} \\
&= -\frac{3+D_1+5D_1^2+D_1^3+2D_1^4}{(1+D_1+D_1^2)^2}
\end{aligned}
$$

hence we have

$$
\begin{aligned}
h(D_1) - P_2^3 &= (P_3^4 - P_3^3)\frac{1+2D_1}{1+D_1} + P_2^4 - P_2^3 \\
&= \frac{1 + 2D_1 + 2D_1^2 + 3D_1^3 + D_1^4 + D_1^5}{D_1(1+D_1+D_1^2)^2}\frac{1+2D_1}{1+D_1} - \frac{3+D_1+5D_1^2+D_1^3+2D_1^4}{(1+D_1+D_1^2)^2} \\
&= \frac{(1 + 2D_1 + 2D_1^2 + 3D_1^3 + D_1^4 + D_1^5)(1+2D_1) - (3+D_1+5D_1^2+D_1^3+2D_1^4)D_1(1+D_1)}{D_1(1+D_1+D_1^2)^2(1+D_1)}
\end{aligned}
$$

The numerator is

$$
\begin{aligned}
& 1 + 2D_1 + 2D_1^2 + 3D_1^3 + D_1^4 + D_1^5 + 2D_1 + 4D_1^2 + 4D_1^3 + 6D_1^4 + 2D_1^5 + 2D_1^6 \\
& -3D_1 - D_1^2 - 5D_1^3 - D_1^4 - 2D_1^5 - 3D_1^2 - D_1^3 - 5D_1^4 - D_1^5 - 2D_1^6 \\
= \; & 1 + D_1 + 2D_1^2 + D_1^3 + D_1^4 \\
= \; & (1+D_1+D_1^2)(1+D_1^2)
\end{aligned}
$$

therefore

$$\begin{aligned} h(D_1) - P_2^3 &= \frac{(1+D_1+D_1^2)(1+D_1^2)}{D_1(1+D_1+D_1^2)^2(1+D_1)} = \frac{1+D_1^2}{D_1(1+D_1)(1+D_1+D_1^2)} \\ &= \frac{1+D_1^2}{D_1+2D_1^2+2D_1^3+D_1^4} > 0 \end{aligned}$$