Technical Appendix for "Information Shocks and Precautionary Saving"

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1 The Model

An agent who lives for $T+1$ periods from 0 to T. He has preferences

$$U = E\left[\sum_{t=0}^{T} \beta^t u(\widetilde{c}_t, \widetilde{m}_t)\right],$$

where

$$u(c,m) = \begin{cases} \frac{1}{1-\gamma}(c-m)^{1-\gamma} & \gamma \neq 1 \\ \ln(c-m) & \gamma = 1 \end{cases} \tag{1}$$

for $\gamma \geq 0$.[1] Then

$$\frac{\partial u}{\partial c}(c,m) = (c-m)^{-\gamma} \tag{2}$$

for all $\gamma \geq 0$.

Let z_t be a sufficient statistic summarizing the information available at t. Given the information vector z_t, the subsistence level of consumption at t will be $m_t(z_t)$, and the consumer will receive income $y_t(z_t)$. The consumer can borrow or lend consumption at the gross interest rate $R > 1$.

Let

$$x_t = y_t + Rb_t \tag{3}$$

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[1] The results here can be simply generalized to the entire hyperbolic absolute risk aversion (HARA) family of preferences. However, analytic results already exist for the most interesting representatives of the HARA class not covered here, i.e. constant absolute risk aversion (CARA) (Caballero (1990)) and quadratic utility, although the preference shocks have to be turned off ($m_t = 0$) in the case of CARA utility.

denote the agent's cash on hand (Deaton (1991)). We can then write the Bellman equation as

$$v_T(x_T, z_T) = u(x_T, m_T(z_T)) \tag{4}$$

and for $0 \leq t < T$,

$$v_t(x_t, z_t) = \max_{b_{t+1}} u(x_t - b_{t+1}, m_t(z_t)) + \beta E_t \left[v_{t+1}(y_{t+1}(\widetilde{z}_{t+1}) + Rb_{t+1}, \widetilde{z}_{t+1}) \right], \tag{5}$$

where E_t is the expectation conditional on z_t being observed at t. Note that we impose no constraints on borrowing. The Bellman equation is solved by the policy function $b_{t+1}(x_t, z_t)$ or, in terms of consumption,

$$c_t(x_t, z_t) = x_t - b_{t+1}(x_t, z_t). \tag{6}$$

Clearly,

$$c_T(x_T, z_T) = x_T. \tag{7}$$

We get from (4) for $t = T$ or from (5) and the Envelope Theorem for $t < T$ that

$$\frac{\partial v_t}{\partial x_t}(x_t, z_t) = (c_t(x_t, z_t) - m_t(z_t))^{-\gamma}. \tag{8}$$

The first-order condition for (5) is

$$(x_t - b_{t+1} - m_t(z_t))^{-\gamma} = \beta R E_t \left[\frac{\partial v_{t+1}}{\partial x_{t+1}} (y_{t+1}(\widetilde{z}_{t+1}) + Rb_{t+1}, \widetilde{z}_{t+1}) \right].$$

Using (6) and (8), this simplifies to the consumption Euler equation

$$\begin{aligned} & (c_t(x_t, z_t) - m_t(z_t))^{-\gamma} \\ = \ & \beta R E_t \left[(c_{t+1}(y_{t+1}(\widetilde{z}_{t+1}) + R(x_t - c_t(x_t, z_t)), \widetilde{z}_{t+1}) - m_{t+1}(\widetilde{z}_{t+1}))^{-\gamma} \right] \end{aligned} \tag{9}$$

It is often more convenient to express consumption functions in terms of wealth, so let us define excess wealth at t by

$$w_t(x_t, z_t) \equiv x_t - m_t(z_t) + \sum_{i=t+1}^{T} \frac{E_t[y_i(\widetilde{z}_i) - m_i(\widetilde{z}_i)]}{R^{i-t}} \tag{10}$$

since, in the absence of uncertainty, excess consumption, $c_t - m_t$, is simply proportional to w_t. If we define

$$x_{t+1}(x_t, z_t, z_{t+1}) = y_{t+1}(z_{t+1}) + R(x_t - c_t(x_t, z_t)), \tag{11}$$

we can define excess wealth next period as

$$w_{t+1}(x_t, z_t, z_{t+1}) \equiv w_{t+1}(x_{t+1}(x_t, z_t, z_{t+1}), z_{t+1}) \tag{12}$$

Finally, we define expected excess wealth next period as

$$W_{t+1}(x_t, z_t) \equiv E_t[w_{t+1}(x_t, z_t, \widetilde{z}_{t+1})]. \tag{13}$$

2

Deriving the Consumption Function

We compute the righthand side of (9) stepwise as follows. For $t < T$, we assume that

$$c_t(x_t, z_t) = A_t x_t + B_t(z_t) + \frac{F_t(z_t)}{x_t} + O\left(\frac{1}{x_t^2}\right), \tag{14}$$

holds for c_{t+1}.

$$\begin{aligned} & c_{t+1}(y_{t+1}(z_{t+1}) + R(x_t - c_t(x_t, z_t)), z_{t+1}) - m_{t+1}(z_{t+1}) \\ = \;& A_{t+1}\left[y_{t+1}(z_{t+1}) + R(x_t - c_t(x_t, z_t))\right] + B_{t+1}(z_{t+1}) - m_{t+1}(z_{t+1}) \\ & + \frac{F_{t+1}(z_{t+1})}{R(x_t - c_t(x_t, z_t))} + O\left(\frac{1}{x_{t+1}^2}\right) \end{aligned}$$

This can be rewritten

$$\begin{aligned} & c_{t+1}(y_{t+1}(z_{t+1}) + R(x_t - c_t(x_t, z_t)), z_{t+1}) - m_{t+1}(z_{t+1}) \\ = \;& A_{t+1} R(x_t - c_t(x_t, z_t)) \\ & \times \left[1 + \frac{A_{t+1} y_{t+1}(z_{t+1}) + B_{t+1}(z_{t+1}) - m_{t+1}(z_{t+1})}{A_{t+1} R(x_t - c_t(x_t, z_t))} \right. \\ & \left. + \frac{F_{t+1}(z_{t+1})}{A_{t+1} R^2 (x_t - c_t(x_t, z_t))^2} + O\left(\frac{1}{x_{t+1}^3}\right)\right]. \end{aligned}$$

Using

$$(1 + x)^{-\gamma} = 1 - \gamma x + \frac{\gamma(\gamma + 1)}{2} x^2, \tag{15}$$

$$\begin{aligned} & (c_{t+1}(y_{t+1}(z_{t+1}) + R(x_t - c_t(x_t, z_t)), z_{t+1}) - m_{t+1}(z_{t+1}))^{-\gamma} \\ = \;& A_{t+1}^{-\gamma} R^{-\gamma} (x_t - c_t(x_t, z_t))^{-\gamma} \\ & \times \left[1 - \gamma \frac{A_{t+1} y_{t+1}(z_{t+1}) + B_{t+1}(z_{t+1}) - m_{t+1}(z_{t+1})}{A_{t+1} R(x_t - c_t(x_t, z_t))} \right. \\ & + \frac{\gamma(\gamma + 1)}{2} \frac{(A_{t+1} y_{t+1}(z_{t+1}) + B_{t+1}(z_{t+1}) - m_{t+1}(z_{t+1}))^2}{A_{t+1}^2 R^2 (x_t - c_t(x_t, z_t))^2} \\ & \left. - \gamma \frac{F_{t+1}(z_{t+1})}{A_{t+1} R^2 (x_t - c_t(x_t, z_t))^2} + O\left(\frac{1}{x_{t+1}^3}\right)\right]. \end{aligned}$$

Taking expectations with respect to $\widetilde{z}_{t+1}$,

$$\begin{aligned}
& E_t\left[(c_{t+1}(y_{t+1}(\widetilde{z}_{t+1})+R(x_t-c_t(x_t,z_t)),\widetilde{z}_{t+1})-m_{t+1}(\widetilde{z}_{t+1}))^{-\gamma}\right] \\
= \;& A_{t+1}^{-\gamma}R^{-\gamma}(x_t-c_t(x_t,z_t))^{-\gamma} \\
& \times\left[1-\gamma\frac{A_{t+1}E_t[y_{t+1}(\widetilde{z}_{t+1})]+E_t[B_{t+1}(\widetilde{z}_{t+1})]-E_t[m_{t+1}(\widetilde{z}_{t+1})]}{A_{t+1}R(x_t-c_t(x_t,z_t))}\right. \\
& +\frac{\gamma(\gamma+1)}{2}\frac{E_t[(A_{t+1}y_{t+1}(\widetilde{z}_{t+1})+B_{t+1}(\widetilde{z}_{t+1})-m_{t+1}(\widetilde{z}_{t+1}))^2]}{A_{t+1}^2R^2(x_t-c_t(x_t,z_t))^2} \\
& \left.-\gamma\frac{E_t[F_{t+1}(\widetilde{z}_{t+1})]}{A_{t+1}R^2(x_t-c_t(x_t,z_t))^2}+O\left(\frac{1}{x_{t+1}^3}\right)\right].
\end{aligned}$$

Using (15) again to raise the expectation by $-1/\gamma$,

$$\begin{aligned}
& \left(E_t\left[(c_{t+1}(y_{t+1}(\widetilde{z}_{t+1})+R(x_t-c_t(x_t,z_t)),\widetilde{z}_{t+1})-m_{t+1}(\widetilde{z}_{t+1}))^{-\gamma}\right]\right)^{-1/\gamma} \\
= \;& A_{t+1}R(x_t-c_t(x_t,z_t)) \\
& \times\left[1+\frac{A_{t+1}E_t[y_{t+1}(\widetilde{z}_{t+1})]+E_t[B_{t+1}(\widetilde{z}_{t+1})]-E_t[m_{t+1}(\widetilde{z}_{t+1})]}{A_{t+1}R(x_t-c_t(x_t,z_t))}+\right. \\
& +\frac{1}{2}\frac{1}{\gamma}\left(\frac{1}{\gamma}+1\right)\frac{\gamma^2(A_{t+1}E_t[y_{t+1}(\widetilde{z}_{t+1})]+E_t[B_{t+1}(\widetilde{z}_{t+1})]-E_t[m_{t+1}(\widetilde{z}_{t+1})])^2}{A_{t+1}^2R^2(x_t-c_t(x_t,z_t))^2} \\
& -\frac{\gamma+1}{2}\frac{E_t[(A_{t+1}y_{t+1}(\widetilde{z}_{t+1})+B_{t+1}(\widetilde{z}_{t+1})-m_{t+1}(\widetilde{z}_{t+1}))^2]}{A_{t+1}^2R^2(x_t-c_t(x_t,z_t))^2} \\
& \left.+\frac{E_t[F_{t+1}(\widetilde{z}_{t+1})]}{A_{t+1}R^2(x_t-c_t(x_t,z_t))^2}+O\left(\frac{1}{x_{t+1}^3}\right)\right].
\end{aligned}$$

This simplifies to

$$\begin{aligned}
& \left(E_t\left[(c_{t+1}(y_{t+1}(\widetilde{z}_{t+1})+R(x_t-c_t(x_t,z_t)),\widetilde{z}_{t+1})-m_{t+1}(\widetilde{z}_{t+1}))^{-\gamma}\right]\right)^{-1/\gamma} \\
= \;& A_{t+1}R(x_t-c_t(x_t,z_t))+A_{t+1}E_t[y_{t+1}(\widetilde{z}_{t+1})]+E_t[B_{t+1}(\widetilde{z}_{t+1})] \\
& -E_t[m_{t+1}(\widetilde{z}_{t+1})]+\frac{E_t[F_{t+1}(\widetilde{z}_{t+1})]}{R(x_t-c_t(x_t,z_t))} \\
& -\frac{\gamma+1}{2}\frac{V_t[A_{t+1}y_{t+1}(\widetilde{z}_{t+1})+B_{t+1}(\widetilde{z}_{t+1})-m_{t+1}(\widetilde{z}_{t+1})]}{A_{t+1}R(x_t-c_t(x_t,z_t))}+O\left(\frac{1}{x_{t+1}^2}\right).
\end{aligned}$$

Substituting this into (9), we get

$$\begin{aligned}
& c_t(x_t,z_t)-m_t(z_t) \\
= \;& (\beta R)^{-1/\gamma}\left\{A_{t+1}R(x_t-c_t(x_t,z_t))+\frac{E_t[F_{t+1}(\widetilde{z}_{t+1})]}{R(x_t-c_t(x_t,z_t))}\right. \\
& +E_t[A_{t+1}y_{t+1}(\widetilde{z}_{t+1})+B_{t+1}(\widetilde{z}_{t+1})-m_{t+1}(\widetilde{z}_{t+1})] \\
& \left.-\frac{\gamma+1}{2}\frac{V_t[A_{t+1}y_{t+1}(\widetilde{z}_{t+1})+B_{t+1}(\widetilde{z}_{t+1})-m_{t+1}(\widetilde{z}_{t+1})]}{A_{t+1}R(x_t-c_t(x_t,z_t))}\right\}+O\left(\frac{1}{x_{t+1}^2}\right).
\end{aligned}$$

We can then solve for c_t:

$$\begin{aligned}
& c_t(x_t, z_t) \\
= \quad & \frac{1}{1+\phi A_{t+1}} \left[m_t(z_t) + \phi \left\{ A_{t+1} x_t + \frac{E_t[A_{t+1} y_{t+1}(\widetilde{z}_{t+1}) - m_{t+1}(\widetilde{z}_{t+1})]}{R} \right.\right. \\
& + \frac{E_t[B_{t+1}(\widetilde{z}_{t+1})]}{R} - \frac{\gamma+1}{2} \frac{V_t[A_{t+1} y_{t+1}(\widetilde{z}_{t+1}) + B_{t+1}(\widetilde{z}_{t+1}) - m_{t+1}(\widetilde{z}_{t+1})]}{A_{t+1} R^2 (x_t - c_t(x_t, z_t))} \\
& \left.\left. + \ \frac{E_t[F_{t+1}(\widetilde{z}_{t+1})]}{R^2(x_t - c_t(x_t, z_t))} \right\} \right] + O\left(\frac{1}{x_{t+1}^2}\right).
\end{aligned}$$

Finally, note that

$$x_t - c_t(x_t, z_t) = \left(1 - \frac{\phi A_{t+1}}{1+\phi A_{t+1}}\right) x_t + O(1) = \frac{x_t}{1+\phi A_{t+1}} + O(1),$$

and

$$x_{t+1} = R(x_t - c_t\,(x_t, z_t)) + O(1) = \frac{R x_t}{1+\phi A_{t+1}} + O(1).$$

Thus,

$$\begin{aligned}
& c_t(x_t, z_t) \\
= \quad & \frac{m_t(z_t)}{1+\phi A_{t+1}} \\
& + \frac{\phi}{1+\phi A_{t+1}} \left\{ A_{t+1} x_t + \frac{E_t[A_{t+1} y_{t+1}(\widetilde{z}_{t+1}) + B_{t+1}(\widetilde{z}_{t+1}) - m_{t+1}(\widetilde{z}_{t+1})]}{R} \right\} \\
& - \phi \frac{\gamma+1}{2} \frac{V_t[A_{t+1} y_{t+1}(\widetilde{z}_{t+1}) + B_{t+1}(\widetilde{z}_{t+1}) - m_{t+1}(\widetilde{z}_{t+1})]}{A_{t+1} R^2 x_t} \\
& + \phi \frac{E_t[F_{t+1}(\widetilde{z}_{t+1})]}{R^2 x_t} + O\left(\frac{1}{x_{t+1}^2}\right).
\end{aligned} \tag{16}$$

Comparing to (14), we obtain the difference equations

$$A_t = \frac{\phi A_{t+1}}{1+\phi A_{t+1}} \tag{17}$$

$$\begin{aligned}
& B_t(z_t) \\
= \quad & \frac{1}{1+\phi A_{t+1}} m_t(z_t) \\
& + \frac{\phi}{1+\phi A_{t+1}} \frac{E_t[A_{t+1} y_{t+1}(\widetilde{z}_{t+1}) + B_{t+1}(\widetilde{z}_{t+1}) - m_{t+1}(\widetilde{z}_{t+1})]}{R}
\end{aligned} \tag{18}$$

$$\begin{aligned}
& F_t(z_t) \\
= \quad & \frac{\phi}{R^2} E_t[F_{t+1}(\widetilde{z}_{t+1})] \\
& - \frac{\phi}{R^2} \frac{\gamma+1}{2A_{t+1}} V_t[A_{t+1} y_{t+1}(\widetilde{z}_{t+1}) + B_{t+1}(\widetilde{z}_{t+1}) - m_{t+1}(\widetilde{z}_{t+1})],
\end{aligned} \tag{19}$$

where we define the conditional variance of a random variable $\widetilde{X}$ by

$$V_t[\widetilde{X}] = E_t[(\widetilde{X} - E_t[\widetilde{X}])^2].$$

Rearranging (17), we obtain the simpler difference equation

$$A_t^{-1} = 1 + \phi^{-1} A_{t+1}^{-1}.$$

Since $A_T = 1$, this has the solution:

$$A_t = (T - t + 1)_{\phi^{-1}}^{-1} = \frac{\phi^{T-t}}{(T - t + 1)_\phi}. \tag{20}$$

Then

$$\begin{aligned}
\frac{1}{1 + \phi A_{t+1}} &= \frac{1}{1 + \phi (T-t)_{\phi^{-1}}^{-1}} = \frac{\phi^{-1}(T-t)_{\phi^{-1}}}{1 + \phi^{-1}(T-t)_{\phi^{-1}}} \\
&= \frac{\phi^{-1}(T-t)_{\phi^{-1}}}{(T-t+1)_{\phi^{-1}}} = \frac{\phi^{-1}\phi^{-(T-t-1)}(T-t)_\phi}{\phi^{-(T-t)}(T-t+1)_\phi} \\
&= \frac{(T-t)_\phi}{(T-t+1)_\phi} = 1 - \frac{\phi^{T-t}}{(T-t+1)_\phi} \\
&= 1 - A_t
\end{aligned} \tag{21}$$

Substituting (17) and (21) into (18), we obtain

$$\begin{aligned}
& B_t(z_t) - m_t(z_t) \\
= \; & A_t \left[\frac{E_t[y_{t+1}(\widetilde{z}_{t+1})]}{R} - m_t(z_t) \right. \\
& \left. + \frac{E_t[B_{t+1}(\widetilde{z}_{t+1})] - E_t[m_{t+1}(\widetilde{z}_{t+1})]}{R A_{t+1}} \right].
\end{aligned}$$

Let us define

$$D_t(z_t) = \frac{B_t(z_t) - m_t(z_t)}{A_t} \tag{22}$$

Then we get the difference equation

$$D_t(z_t) = \frac{E_t[y_{t+1}(\widetilde{z}_{t+1})]}{R} - m_t(z_t) + \frac{1}{R} E_t[D_{t+1}(\widetilde{z}_{t+1})]. \tag{23}$$

Since $D_T = 0$, (23) has the solution

$$D_t(z_t) = \sum_{i=t+1}^{T} \frac{E_t[y_i(\widetilde{z}_i) - m_i(\widetilde{z}_i)]}{R^{i-t}} - m_t(z_t). \tag{24}$$

Then we get

$$
\begin{aligned}
& B_t(z_t) \\
= \quad & m_t(z_t) + \frac{\phi^{T-t}}{(T-t+1)_\phi} \left[-m_t(z_t) + \sum_{i=t+1}^{T} \frac{E_t[y_i(\tilde{z}_i) - m_i(\tilde{z}_i)]}{R^{i-t}} \right].
\end{aligned}
$$

Finally, substituting (22) into (19), we get

$$
F_t = \frac{\phi}{R^2} \left(E_t[F_{t+1}(\tilde{z}_{t+1})] - \frac{\gamma+1}{2} A_{t+1} V_t[y_{t+1}(\tilde{z}_t) + D_{t+1}(\tilde{z}_{t+1})] \right),
$$

noting that A_{t+1}^2 comes out of the variance when we factor out the A_{t+1} from inside. Using (24), this becomes

$$
\begin{aligned}
& F_t(z_t) \\
= \quad & \frac{\phi}{R^2} \left[E_t[F_{t+1}(\tilde{z}_{t+1})] - \frac{\gamma+1}{2} A_{t+1} V_t \left[\sum_{i=t+1}^{T} \frac{E_{t+1}[y_i(\tilde{z}_i) - m_i(\tilde{z}_i)]}{R^{i-t-1}} \right] \right].
\end{aligned}
$$

Notice that, absent a growth trend in y_t and m_t, in the limit as $t \to -\infty$ (or equivalently as $T \to \infty$), this difference equation will only have a convergent solution if $\phi/R^2 < 1$, as was noted in Feigenbaum (2005). Regardless, for finite t, the difference equation has the solution

$$
F_t(z_t) = -\frac{\gamma+1}{2} \sum_{i=t+1}^{T} \left(\frac{\phi}{R^2} \right)^{i-t} A_i E_t \left[V_{i-1} \left[\sum_{j=i}^{T} \frac{E_i[y_j(\tilde{z}_j) - m_j(\tilde{z}_j)]}{R^{j-i}} \right] \right].
$$

Substituting in (20), this simplifies to

$$
F_t(z_t) = -\frac{\gamma+1}{2} \sum_{i=t+1}^{T} \frac{\phi^{T-t}}{(T-i+1)_\phi} E_t \left[V_{i-1} \left[\sum_{j=i}^{T} \frac{E_i[y_j(\tilde{z}_j) - m_j(\tilde{z}_j)]}{R^{j-t}} \right] \right].
$$

Incorporating these solutions into (14), we obtain the following expression for the consumption function:

$$
\begin{aligned}
& c_t(x_t, z_t) - m_t(z_t) \\
= \quad & \frac{\phi^{T-t}}{(T-t+1)_\phi} \left[x_t - m_t(z_t) + \sum_{i=t+1}^{T} \frac{E_t[y_i(\tilde{z}_i) - m_i(\tilde{z}_i)]}{R^{i-t}} \right] \\
& - \frac{\gamma+1}{2} \sum_{i=t+1}^{T} \frac{\phi^{T-t}}{(T-i+1)_\phi} \\
& \times E_t \left[V_{i-1} \left[\sum_{j=i}^{T} \frac{E_i[y_j(\tilde{z}_j) - m_j(\tilde{z}_j)]}{R^{j-t}} \right] \right] \frac{1}{x_t} + O\left(\frac{1}{x_t^2} \right) \qquad (25)
\end{aligned}
$$

1 Deriving the Growth Rate

To begin, we wish to work out an expression for

$$c_{t+1}(x_t, z_t, z_{t+1}) = c_{t+1}(x_{t+1}(x_t, z_t), z_{t+1}). \tag{26}$$

Updating (25) to $t+1$, we get

$$\begin{aligned} & c_{t+1}(x_{t+1}, z_{t+1}) - m_{t+1}(z_{t+1}) \\ = \quad & \frac{\phi^{T-t-1}}{(T-t)_\phi} \left[x_{t+1} - m_{t+1}(z_{t+1}) + \sum_{i=t+2}^{T} \frac{E_{t+1}[y_i(\widetilde{z}_i) - m_i(\widetilde{z}_i)]}{R^{i-t-1}} \right] \\ & - \frac{\gamma+1}{2} \sum_{i=t+2}^{T} \frac{\phi^{T-t-1}}{(T-i+1)_\phi} \\ & \times E_{t+1} \left[V_{i-1} \left[\sum_{j=i}^{T} \frac{E_i[y_j(\widetilde{z}_j) - m_j(\widetilde{z}_j)]}{R^{j-t-1}} \right] \right] \frac{1}{x_{t+1}} + O\left(\frac{1}{x_{t+1}^2} \right). \end{aligned} \tag{27}$$

Plugging (25) into (11), we get

$$\begin{aligned} & x_{t+1}(x_t, z_t, z_{t+1}) - m_{t+1}(z_{t+1}) \\ = \quad & y_{t+1}(z_{t+1}) - m_{t+1}(z_{t+1}) - E_t\left[y_{t+1}(\widetilde{z}_{t+1}) - m_{t+1}(\widetilde{z}_{t+1}) \right] \\ & + R \frac{(T-t)_\phi}{(T-t+1)_\phi} \left[x_t - m_t(z_t) + \frac{E_t[y_{t+1}(\widetilde{z}_{t+1}) - m_{t+1}(\widetilde{z}_{t+1})]}{R} \right] \\ & - \frac{R\phi^{T-t}}{(T-t+1)_\phi} \sum_{i=t+2}^{T} \frac{E_t[y_i(\widetilde{z}_i) - m_i(\widetilde{z}_i)]}{R^{i-t}} \\ & + R \frac{\gamma+1}{2} \sum_{i=t+1}^{T} \frac{\phi^{T-t}}{(T-i+1)_\phi} \\ & \times E_t \left[V_{i-1} \left[\sum_{j=i}^{T} \frac{E_i[y_j(\widetilde{z}_j) - m_j(\widetilde{z}_j)]}{R^{j-t}} \right] \right] \frac{1}{x_t} \Bigg\} + O\left(\frac{1}{x_t^2} \right) \end{aligned} \tag{28}$$

Note that this implies

$$\frac{1}{x_t} = R \frac{(T-t)_\phi}{(T-t+1)_\phi} \frac{1}{x_{t+1}(x_t, z_t, z_{t+1})} + O\left(\frac{1}{x_t^2} \right) \tag{29}$$

From (28) and (29), we get excess wealth next period is

$$\begin{aligned}
& w_{t+1}(x_t, z_t, z_{t+1}) \\
= \; & y_{t+1}(z_{t+1}) - m_{t+1}(z_{t+1}) - E_t\left[y_{t+1}(\widetilde{z}_{t+1}) - m_{t+1}(\widetilde{z}_{t+1})\right] \\
& + R\frac{(T-t)_\phi}{(T-t+1)_\phi}\left\{\left[x_t - m_t(z_t) + \sum_{i=t+1}^{T}\frac{E_t[y_i(\widetilde{z}_i) - m_i(\widetilde{z}_i)]}{R^{i-t}}\right]\right. \\
& + R\frac{\gamma+1}{2}\sum_{i=t+1}^{T}\frac{\phi^{T-t}}{(T-i+1)_\phi} \\
& \left.\times\, E_t\left[V_{i-1}\left[\sum_{j=i}^{T}\frac{E_i[y_j(\widetilde{z}_j) - m_j(\widetilde{z}_j)]}{R^{j-t}}\right]\right]\frac{1}{x_{t+1}(x_t, z_t, z_{t+1})}\right\} \\
& + O\left(\frac{1}{x_t^2}\right). \qquad (30)
\end{aligned}$$

Then expected excess wealth is

$$\begin{aligned}
& W_{t+1}(x_t, z_t) \\
= \; & R\frac{(T-t)_\phi}{(T-t+1)_\phi}\left\{x_t - m_t(z_t) + \sum_{i=t+1}^{T}\frac{E_t[y_i(\widetilde{z}_i) - m_i(\widetilde{z}_i)]}{R^{i-t}}\right. \\
& + R\frac{\gamma+1}{2}\sum_{i=t+1}^{T}\frac{\phi^{T-t}}{(T-i+1)_\phi} \\
& \left.\times\, E_t\left[V_{i-1}\left[\sum_{j=i}^{T}\frac{E_i[y_j(\widetilde{z}_j) - m_j(\widetilde{z}_j)]}{R^{j-t}}\right]\right]\frac{1}{x_{t+1}(x_t, z_t, z_{t+1})}\right\} \\
& + O\left(\frac{1}{x_t^2}\right). \qquad (31)
\end{aligned}$$

Note that

$$w_{t+1}(x_t, z_t, z_{t+1}) = W_{t+1}(x_t, z_t) + O(1) = x_{t+1}(x_t, z_t, z_{t+1}) + O(1),$$

so

$$\begin{aligned}
\frac{1}{w_{t+1}(x_t, z_t, z_{t+1})} &= \frac{1}{W_{t+1}(x_t, z_t)} + O\left(\frac{1}{x_t^2}\right) \\
&= \frac{1}{x_{t+1}(x_t, z_t, z_{t+1})} + O\left(\frac{1}{x_t^2}\right). \qquad (32)
\end{aligned}$$

Applying (12) and (32) to (27), we obtain the expression

$$\begin{aligned}
& c_{t+1}(x_t, z_t, z_{t+1}) - m_{t+1}(z_{t+1}) \\
= \; & \frac{\phi^{T-t-1}}{(T-t)_\phi} w_{t+1}(x_t, z_t, z_{t+1}) \\
& -R^2 \frac{\gamma+1}{2} \sum_{i=t+2}^{T} \frac{\phi^{T-t-1}}{(T-i+1)_\phi} \\
& \times E_{t+1}\left[V_{i-1}\left[\sum_{j=i}^{T} \frac{E_i[y_j(\tilde{z}_j) - m_j(\tilde{z}_j)]}{R^{j-t}} \right]\right] \frac{1}{W_{t+1}(x_t, z_t)} + O\left(\frac{1}{x_t^2}\right).
\end{aligned}$$

Isolating the variance and higher order terms,

$$\begin{aligned}
& c_{t+1}(x_t, z_t, z_{t+1}) - m_{t+1}(z_{t+1}) \\
= \; & \frac{\phi^{T-t-1}}{(T-t)_\phi} w_{t+1}(x_t, z_t, z_{t+1}) \\
& \times \left[1 - R^2 \frac{\gamma+1}{2} \sum_{i=t+2}^{T} \frac{(T-t)_\phi}{(T-i+1)_\phi} \right. \\
& \left. \times \; E_{t+1}\left[V_{i-1}\left[\sum_{j=i}^{T} \frac{E_i[y_j(\tilde{z}_j) - m_j(\tilde{z}_j)]}{R^{j-t}} \right]\right] \frac{1}{W_{t+1}^2(x_t, z_t)} + O\left(\frac{1}{x_t^3}\right) \right].
\end{aligned} \tag{33}$$

Using (31), we can also rewrite (25) in terms of W_{t+1}:

$$\begin{aligned}
& c_t(x_t, z_t) - m_t(z_t) \\
= \; & \frac{\phi^{T-t}}{R(T-t)_\phi} W_{t+1}(x_t, z_t) \\
& -\frac{\phi^{T-t}}{(T-t+1)_\phi} R \frac{\gamma+1}{2} \sum_{i=t+1}^{T} \frac{\phi^{T-t}}{(T-i+1)_\phi} \\
& \times E_t\left[V_{i-1}\left[\sum_{j=i}^{T} \frac{E_i[y_j(\tilde{z}_j) - m_j(\tilde{z}_j)]}{R^{j-t}} \right]\right] \frac{1}{W_{t+1}(x_t, z_t)} \\
& -\frac{\gamma+1}{2} \sum_{i=t+1}^{T} \frac{\phi^{T-t}}{(T-i+1)_\phi} \\
& \times E_t\left[V_{i-1}\left[\sum_{j=i}^{T} \frac{E_i[y_j(\tilde{z}_j) - m_j(\tilde{z}_j)]}{R^{j-t}} \right]\right] R \frac{(T-t)_\phi}{(T-t+1)_\phi} \frac{1}{W_{t+1}(x_t, z_t)} \\
& +O\left(\frac{1}{x_t^2}\right)
\end{aligned}$$

$$
\begin{aligned}
& c_t(x_t, z_t) - m_t(z_t) \\
= \quad & \frac{\phi^{T-t}}{R(T-t)_\phi} W_{t+1}(x_t, z_t) \\
& \times \left[1 - R^2 \frac{\gamma+1}{2} \sum_{i=t+1}^{T} \frac{(T-t)_\phi}{(T-i+1)_\phi} \right. \\
& \left. E_t \left[V_{i-1} \left[\sum_{j=i}^{T} \frac{E_i[y_j(\widetilde{z}_j) - m_j(\widetilde{z}_j)]}{R^{j-t}} \right] \right] \frac{1}{W_{t+1}^2(x_t, z_t)} \right] + O\left(\frac{1}{x_t^2}\right) \quad (34)
\end{aligned}
$$

Combining (33) and (34), we get

$$
\begin{aligned}
& \frac{c_{t+1}(x_t, z_t, z_{t+1}) - m_t(z_{t+1})}{c_t(x_t, z_t) - m_t(z_t)} \\
= \quad & G_0 \frac{w_{t+1}(x_t, z_t, z_{t+1})}{W_{t+1}(x_t, z_t)} \\
& \times \left\{ 1 + \frac{\gamma+1}{2} V_t \left[\sum_{j=t+1}^{T} \frac{E_{t+1}[y_j(\widetilde{z}_j) - m_j(\widetilde{z}_j)]}{R^{j-t-1}} \right] \frac{1}{W_{t+1}^2(x_t, z_t)} \right. \\
& + R^2 \frac{\gamma+1}{2} \sum_{i=t+2}^{T} \frac{(T-t)_\phi}{(T-i+1)_\phi} \\
& \times (E_t - E_{t+1}) \left[V_{i-1} \left[\sum_{j=i}^{T} \frac{E_i[y_j(\widetilde{z}_j) - m_j(\widetilde{z}_j)]}{R^{j-t}} \right] \right] \frac{1}{W_{t+1}^2(x_t, z_t)} \\
& \left. \times + O\left(\frac{1}{x_t^3}\right) \right\} \qquad (35)
\end{aligned}
$$

2 Some Simple Examples

Let us consider how the second-order consumption function behaves in several simple examples. In most of these examples, we assume z_t follows a 2-state Markov process. Assuming a symmetric transition matrix, this matrix can be parameterized as

$$
\Pi = \begin{bmatrix} 1-\lambda & \lambda \\ \lambda & 1-\lambda \end{bmatrix}, \qquad (36)
$$

where $\lambda \in [0, 1]$. Note that powers of the transition matrix are given by

$$
\Pi^n = \frac{1}{2} \begin{bmatrix} 1 + (1-2\lambda)^n & 1 - (1-2\lambda)^n \\ 1 - (1-2\lambda)^n & 1 + (1-2\lambda)^n \end{bmatrix}. \qquad (37)
$$

Finally, we assume the initial distribution is the invariant distribution of Π, so

$$p_t = p_0 = \left[\begin{array}{cc} \frac{1}{2} & \frac{1}{2} \end{array}\right]. \tag{38}$$

Given this invariant distribution, we have

$$\begin{aligned} E[\widetilde{z}_t] &= \frac{Z_1 + Z_2}{2} \\ V[\widetilde{z}_t] &= \frac{1}{2}\left[Z_1 - \frac{Z_1 + Z_2}{2}\right]^2 + \frac{1}{2}\left[Z_2 - \frac{Z_1 + Z_2}{2}\right]^2 = \frac{(Z_2 - Z_1)^2}{4} \end{aligned}$$

$$\begin{aligned} &\operatorname{cov}(\widetilde{z}_t, \widetilde{z}_{t+1}) \\ = &\ \frac{1}{2}\left[(1-\lambda)\left(Z_1 - \frac{Z_1 + Z_2}{2}\right)^2 + \lambda\left(Z_1 - \frac{Z_1 + Z_2}{2}\right)\left(Z_2 - \frac{Z_1 + Z_2}{2}\right)\right] \\ &+ \frac{1}{2}\left[(1-\lambda)\left(Z_2 - \frac{Z_1 + Z_2}{2}\right)^2 + \lambda\left(Z_1 - \frac{Z_1 + Z_2}{2}\right)\left(Z_2 - \frac{Z_1 + Z_2}{2}\right)\right] \\ = &\ (1 - 2\lambda)\frac{(Z_2 - Z_1)^2}{4}. \end{aligned}$$

Thus, the lag-1 autocorrelation of z_t is $1 - 2\lambda$.

2.1 The Timing of an Income Shock

To begin with, let us consider a generalization of the thought experiment studied by Blundell and Stoker (1999). They considered a three-period model in which a stochastic income was received either in the second period or the third period, with the variance of the income being the same in either case, and showed that precautionary effects would be stronger if the stochastic income was received at an earlier time.

Here, we fix a mean income process $\mu_t \geq 0$ for $t = 0, \ldots, T$, and suppose that $Z_i = (-1)^i$ for $i = 1, 2$. Let $0 < s \leq T_W$, and consider a model in which the income process is then

$$y_t(z_t) = \begin{cases} \mu_t & t \neq s \\ \mu_s + \sigma z_t & t = s \end{cases} \tag{39}$$

for $\sigma < \mu_s$. For this experiment, we assume z_t is i.i.d. with $\lambda = 1/2$ in the transition matrix (36). Thus, although z_t is observable for $t < s$, it confers no information about the stochastic income y_s. For $t < s$, $E_t[y_s(\widetilde{z}_s)] = \mu_s$ and $V_{t-1}[E_t[y_s(\widetilde{z}_s)]] = 0$ while $E_s[y_s(\widetilde{z}_s)] = y_s(z_s)$ and $V_{s-1}[E_s[y_s(\widetilde{z}_s)]] = \sigma^2$. Then for $q \leq t < s$, $E_q[V_{t-1}[E_t[y_s(\widetilde{z}_s)]]] = 0$, and $E_q[V_{s-1}[E_s[y_s(\widetilde{z}_s)]]] = \sigma^2$.

Substituting these moments into (25), we get for $t < s$,

$$\begin{aligned} c_t(x_t, z_t) &= \frac{\phi^{T-t}}{(T-t+1)_\phi}\left[x_t + \sum_{i=t+1}^{T}\frac{\mu_i}{R^{i-t}}\right] - \frac{\gamma+1}{2}\frac{\phi^{T-t}}{(T-s+1)_\phi}\frac{\sigma^2}{R^{2(s-t)}}\frac{1}{x_t} \\ &\quad + O\left(\frac{1}{x_t^2}\right). \end{aligned} \tag{40}$$

Notice that only the second term depends on the timing of the income shock s. This term, proportional to the variance of the income shock, represents the decrease in consumption that occurs as a consequence of precautionary saving, so we can define precautionary saving[2] as

$$P_t(x_t) = \frac{\gamma+1}{2}\frac{\phi^{T-t}}{(T-s+1)_\phi}\frac{\sigma^2}{R^{2(s-t)}}\frac{1}{x_t}. \tag{41}$$

If we increase s, the effect on precautionary saving and consumption is ambiguous. On the one hand, $P_t(x_t)$ is proportional to $R^{-2(s-t)}$. This reflects the fact that the present value of future income is discounted, and this discount factor gets squared after taking the variance. It is this effect that Blundell and Stoker (1999) emphasize. On the other hand, there is the factor of $(T-s+1)_\phi^{-1}$, which reflects the fact that there are $T-s+1$ periods after the shock is realized (including the period when it happens) over which to distribute the effects of the shock. As s increases, there are fewer periods over whch to distribute these shocks, so the effect of the shock gets larger. This is the effect that Eeckhoudt, Gollier, and Treich (EGT) (2005) emphasize.

In general, $P_t(x_t)$ will decrease when s increases from s_1 to s_2 iff

$$\frac{1}{(T-s_1+1)_\phi R^{2(s_1-t)}} > \frac{1}{(T-s_2+1)_\phi R^{2(s_2-t)}},$$

or equivalently if

$$\frac{\phi^{T-s_1+1}-1}{\phi^{T-s_2+1}-1} < R^{2(s_2-s_1)}. \tag{42}$$

For the special case where $T = 2$, $s_1 = 1$, and $s_2 = 2$, this condition reduces to

$$\frac{\phi^2-1}{\phi-1} = \phi + 1 < R^2. \tag{43}$$

Blundell and Stoker considered a calibrated model where a period corresponded to a year. If $\beta R = 1$, with annual interest rates on the order of 3%, $\phi = R \sim 2$. In that case, the condition (43) will be satisfied, and, indeed, precautionary

[2]Note that this only accounts for saving at time t in response to future uncertainty. There may be saving accrued at t that has accumulated in response to precautionary saving in earlier periods, but this will be incorporated within current cash on hand x_t.

saving will decrease as s increases. On the other hand, if we view a period as corresponding to a year and calibrate β and R close to 1, the condition (43) will not be satisfied, and EGT point out that the Blundell-Stoker result would be reversed if a smaller R was chosen.

In the limit of large T (under which we must have $\phi > 1$), the lefthand side of (42) converges to $\phi^{s_2-s_1}$. Thus, in this limit, precautionary saving will decrease as s increases from s_1 to s_2 if $\phi > R^2$, which is also the necessary condition that Feigenbaum (2005) found in order for the second-order approximation to work in the limit of large T. Thus, for longer time horizons, the Blunder and Stoker result, that precautionary saving is a decreasing function of s, prevails.

Finally, note that for this thought experiment where z_t is i.i.d., and the income shock is revealed when it is received, Skinner's (1988) formula

$$\frac{c_{t+1}(x_t, z_t, \widetilde{z}_{t+1})}{c_t(x_t, z_t)} = G_0 \left[1 + \frac{\gamma+1}{2} \frac{V_t[y_{t+1}(\widetilde{z}_{t+1}]}{W_{t+1}^2(x_t, z_t)} + O\left(\frac{1}{x_t^3}\right)\right] \frac{w_{t+1}(x_t, z_t, \widetilde{z}_{t+1})}{W_{t+1}(x_t, z_t)}. \tag{44}$$

for the expected rate of consumption growth is still valid. From $s-1$ to s, the expected rate of consumption growth is

$$\begin{aligned} & \frac{E_{s-1}[c_s(x_{s-1}, z_{s-1}, \widetilde{z}_s)]}{c_{s-1}(x_{s-1}, z_{s-1})} \\ = \quad & G_0 \left[1 + \frac{\gamma+1}{2} \frac{\sigma^2}{W_s^2(x_{s-1}, z_{s-1})} + O\left(\frac{1}{x_{s-1}^3}\right)\right]. \end{aligned} \tag{45}$$

For all other t,

$$\frac{E_t[c_{t+1}(x_t, z_t, \widetilde{z}_{t+1})]}{c_t(x_t, z_t)} = G_0 + O\left(\frac{1}{x_t^3}\right). \tag{46}$$

For a given x_t, consumption at $t < s$ will be lower with uncertainty than it would be without uncertainty, but this is a level effect. The rate of consumption growth will only change between $s-1$ and s.

2.2 The Timing of an Information Shock

EGT (2005) proposed another thought experiment that isolates the effect of how much time there is after the shock. They again consider a three-period model. The income that is stochastic is always received in the third period, but they explore what happens if the information about the third-period income is revealed in the second period as opposed to the third period. Their main result is that revealing the income shock at an earlier time always leads to a reduction in precautionary saving.

Here, we consider a generalization of this experiment. Let $0 \leq s_1 \leq s_2 = T_W$ and suppose that the income process is given by (39), where $s = s_2$. Now, however, the z_t process is such that z_t is i.i.d with equal probability of being 0 or 1 for $t \le s_1$. However, $z_t = z_{s_1}$ for $t \ge s_1$. Thus, the stochastic income is revealed at s_2, but the information shock that reveals what the income will be occurs at $s_1 \le s_2$. Then

$$E_t[y_{s_2}(\widetilde{z}_{s_2})] = \begin{cases} \mu_{s_2} & t < s_1 \\ \mu_{s_2} + \sigma z_{s_1} & t \ge s_1 \end{cases} .$$

$$V_{t-1}[E_t[y_{s_2}(\widetilde{z}_{s_2})]] = \begin{cases} 0 & t < s_1 \\ \sigma^2 & t = s_1 \\ 0 & t > s_1 \end{cases}$$

and for $q \le t-1$, $E_q[V_{t-1}[E_t[y_{s_2}(\widetilde{z}_{s_2})]]] = V_{t-1}[E_t[y_{s_2}(\widetilde{z}_{s_2})]]$. Substituting these moments into (25), for $t < s_1$,

$$\begin{aligned} c_t(x_t, z_t) &= \frac{\phi^{T-t}}{(T-t+1)_\phi}\left[x_t + \sum_{i=t+1}^{T} \frac{\mu_i}{R^{i-t}}\right] \\ &\quad - \frac{\gamma+1}{2}\frac{\phi^{T-t}}{(T-s_1+1)_\phi}\frac{\sigma^2}{R^{2(s_2-t)}}\frac{1}{x_t} + O\left(\frac{1}{x_t^2}\right). \end{aligned}$$

Since $(n)_\phi$ is an increasing function of n, $c_t(x_t, z_t)$ is a decreasing function of s_1. Thus, if the shock is revealed at a later time, precautionary saving increases.

Now let us consider how expected consumption growth behaves in this experiment. Skinner's formula (44) does not deal with situations like this where the information about a stochastic income is revealed before the income is earned, but the generalized formula (35) does. For $t \ne s_1 - 1$, consumption growth from t to $t+1$ is again $(\beta R)^{1/\gamma}$ to second-order in x_t^{-1}. However, from $s_1 - 1$ to s_1, the expected rate of consumption growth is

$$\begin{aligned} &\frac{E_{s_1-1}[c_{s_1}(x_{s_1-1}, z_{s_1-1}, \widetilde{z}_{s_1})]}{c_{s_1-1}(x_{s_1-1}, z_{s_1-1})} \\ &= G_0\left[1 + \frac{\gamma+1}{2}\frac{\sigma^2}{R^{2(s_2-s_1)}W_{s_1}^2(x_{s_1-1}, z_{s_1-1})} + O\left(\frac{1}{x_{s_1-1}^3}\right)\right], \end{aligned} \quad (47)$$

and the rate of consumption growth between the periods when the information about the stochastic income is revealed does increase. Consistent with the finding above that precautionary saving decreases as s_1 moves forward in time, the increase in consumption growth from $s_1 - 1$ to s_1 also decreases as the time of the information shock moves forward.

2.3 Markov Information Shocks

Suppose that the income process satisfies (39) where $z_t = \pm 1$ again, but now z_t is a Markov process with transition matrix (36). In the special case when $\lambda = 1/2$, the experiment is the same as in Section 2.1. However, for $\lambda \ne 1/2$, the information shock at $t < s$ conveys useful information about y_s.

For $t < s$, the expectation of y_s is

$$\begin{bmatrix} E[y_s(\widetilde{z}_s)|z_t = -1] \\ E[y_s(\widetilde{z}_s)|z_t = 1] \end{bmatrix} = \Pi^{s-t} \begin{bmatrix} \mu_s - \sigma \\ \mu_s + \sigma \end{bmatrix} = \begin{bmatrix} \mu_s - (1-2\lambda)^{s-t}\sigma \\ \mu_s + (1-2\lambda)^{s-t}\sigma \end{bmatrix}.$$

Let Π_i denote the ith row of the transition matrix Π. For $t \leq s$, the variance of $E_t[\widetilde{y}_s]$ conditional on $z_{t-1} = -1$ is

$$V[E_t[\widetilde{y}_s]|z_{t-1} = -1] = E[(E_t[\widetilde{y}_s] - E[\widetilde{y}_s|z_{t-1} = -1])^2|z_{t-1} = -1]$$

$$\begin{aligned} &= \Pi_1 \begin{bmatrix} \left([\mu_s - (1-2\lambda)^{s-t}\sigma] - \left[\mu_s - (1-2\lambda)^{s-t+1}\sigma\right]\right)^2 \\ \left([\mu_s + (1-2\lambda)^{s-t}\sigma] - \left[\mu_s - (1-2\lambda)^{s-t+1}\sigma\right]\right)^2 \end{bmatrix} \\ &= \sigma^2 \begin{bmatrix} 1-\lambda & \lambda \end{bmatrix} \begin{bmatrix} \left((1-2\lambda)^{s-t+1} - (1-2\lambda)^{s-t}\right)^2 \\ \left((1-2\lambda)^{s-t+1} + (1-2\lambda)^{s-t}\right)^2 \end{bmatrix} \\ &= 4\lambda(1-\lambda)(1-2\lambda)^{2(s-t)}\sigma^2 \end{aligned}$$

Likewise, for $z_{t-1} = 1$,

$$E[(E_t[\widetilde{y}_s] - E[\widetilde{y}_s|z_{t-1} = 1])^2|z_{t-1} = 1]$$

$$\begin{aligned} &= \Pi_2 \begin{bmatrix} \left([\mu_s - (1-2\lambda)^{s-t}\sigma] - \left[\mu_s + (1-2\lambda)^{s-t+1}\sigma\right]\right)^2 \\ \left([\mu_s + (1-2\lambda)^{s-t}\sigma] - \left[\mu_s + (1-2\lambda)^{s-t+1}\sigma\right]\right)^2 \end{bmatrix} \\ &= 4\lambda(1-\lambda)(1-2\lambda)^{2(s-t)}\sigma^2. \end{aligned}$$

Thus,

$$V_{t-1}\left[E_t[\widetilde{y}_s]\right] = 4\lambda(1-\lambda)(1-2\lambda)^{2(s-t)}\sigma^2, \tag{48}$$

and for $u \leq t-1 < s$,

$$E_u\left[V_{t-1}\left[E_t[\widetilde{y}_s]\right]\right] = 4\lambda(1-\lambda)(1-2\lambda)^{2(s-t)}\sigma^2. \tag{49}$$

Substituting (49) into (25), we get

$$\begin{aligned} & c_t(x_t, z_t) \\ = \;& \frac{\phi^{T-t}}{(T-t+1)_\phi}\left[x_t + \sum_{i=t+1}^{T}\frac{\mu_i}{R^{i-t}} + \frac{(1-2\lambda)^{s-t}\sigma z_t}{R^{s-t}}\right] \\ & -\frac{\gamma+1}{2}\frac{\phi^{T-t}4\lambda(1-\lambda)}{R^{2(s-t)}}\sum_{i=t+1}^{s}\frac{(1-2\lambda)^{2(s-i)}}{(T-i+1)_\phi}\frac{\sigma^2}{x_t} + O\left(\frac{1}{x_t^2}\right) \end{aligned} \tag{50}$$

If we define

$$H(p,q) = \sum_{i=0}^{p-1}\frac{(1-2\lambda)^{2i}}{(q-p+i+1)_\phi}, \tag{51}$$

this can be rewritten as

$$\begin{aligned} c_t(x_t, z_t) &= \frac{\phi^{T-t}}{(T-t+1)_\phi}\left[x_t + \sum_{i=t+1}^{T}\frac{\mu_i}{R^{i-t}} + (1-2\lambda)^{s-t}\sigma z_t\right] \\ &\quad -2(\gamma+1)\phi^{T-t}\frac{\lambda(1-\lambda)}{R^{2(s-t)}}H(s-t,T-t)\frac{\sigma^2}{x_t} + O\left(\frac{1}{x_t^2}\right). \end{aligned}$$

If we assume the initial distribution has equal probability for $z_0 = 0$ or $z_0 = 1$, the unconditional expectation of c_t is

$$\begin{aligned} E[c_t(x_t, \tilde{z}_t)] &= \frac{\phi^{T-t}}{(T-t+1)_\phi}\left[x_t + \sum_{i=t+1}^{T}\frac{\mu_i}{R^{i-t}}\right] \qquad (52) \\ &\quad -2(\gamma+1)\phi^{T-t}\frac{\lambda(1-\lambda)}{R^{2(s-t)}}H(s-t,T-t)\frac{\sigma^2}{x_t} + O\left(\frac{1}{x_t^2}\right). \end{aligned}$$

The second term corresponds to the effects of precautionary saving. The $R^{-2(s-t)}$ factor in the precautionary term represents the effect of discounting. If we delay the time s of the income shock, discounting reduces the precautionary term. On the other hand, $H(s-t, T-t)$ decreases with s since $H(p,q)$ decreases with p. As can be seen in (51), if we increase p, this decreases the denominators of the fractions being summed. This captures the effect already seen in the previous thought experiments that if a shock is later then there will be less time afterwards to distribute the effects of the shock. In addition, increasing p also increases the number of fractions to be summed, reflecting the fact that each information shock causes precautionary saving. If the income shock is later, there will be more information shocks. Thus, the effect of delaying the income shock on precautionary saving is ambiguous.

Now let us consider the expected rate of consumption growth for $t < s$. From Eq. (35),

$$\begin{aligned} &\frac{E_t[c_{t+1}(x_t, z_t, \tilde{z}_{t+1})]}{c_t(x_t, z_t)} \\ &= G_0\left[1 + \frac{\gamma+1}{2}V_t\left[\frac{E_{t+1}[y_s(\tilde{z}_s)]}{R^{s-t-1}}\right]\frac{1}{W_{t+1}^2(x_t, z_t)} + O\left(\frac{1}{x_t^3}\right)\right]. \end{aligned}$$

Substituting in the moment (48), we get for $t < s$ that the increase in the growth rate above the perfect-foresight value of G_0 is

$$\begin{aligned} \Delta G_t &= \frac{E_t[c_{t+1}(x_t, z_t, \tilde{z}_{t+1})]}{c_t(x_t, z_t)} - G_0 \\ &= 2(\gamma+1)\left(\frac{1-2\lambda}{R}\right)^{2(s-t-1)}\frac{\lambda(1-\lambda)\sigma^2}{W_{t+1}^2(x_t, z_t)}G_0 + O\left(\frac{1}{x_t^3}\right). \qquad (53) \end{aligned}$$

If we assume $\lambda < 1/2$, so z_t is positively autocorrelated then information is revealed about y_s at every period before s, so ΔG_t is always positive for $t < s$.

Since $(1-2\lambda)/R < 1$, if we hold expected wealth fixed at $W_{t+1} = W$, ΔG_t decays exponentially in the time to the income shock $s-t$ with a decay factor proportional to the autocorrelation of z_t. Thus, the greater the correlation in z_t is the slower the rate of decay will be. That said, a high autocorrelation will also imply a small value of λ, which will suppress the effects of precautionary saving at all $t < s$.

2.4 An AR(1) Income Process

Now let us suppose instead that income follows an AR(1) income process like in Huggett (1996). For $t = 0, ..., T$, income is

$$y_t = \mu_t + z_t,$$

where μ_t is known from birth and z_t represents an AR(1) income shock. Let ε_t for $t = 0, ..., T$ be independent, mean-zero noise variables with finite moments such that

$$V[\widetilde{\varepsilon}_t] = \sigma_\varepsilon^2.$$

The income innovations then obey

$$z_0 = \varepsilon_0$$

and, for $t > 0$,

$$z_t = \rho z_{t-1} + \varepsilon_t,$$

where $|\rho| < 1$.

Then

$$E[\widetilde{z}_{t+1}|z_t] = E[\rho z_t + \widetilde{\varepsilon}_{t+1}|z_t] = \rho z_t.$$

Suppose for $s \geq 0$ that

$$E[\widetilde{z}_{t+s}|z_t] = \rho^s z_t.$$

Then

$$E[\widetilde{z}_{t+s+1}|z_t] = E[\rho \widetilde{z}_{t+s} + \widetilde{\varepsilon}_{t+s+1}|z_t] = \rho E[\widetilde{z}_{t+s}|z_t] = \rho^{s+1} z_t.$$

$$V[E[\widetilde{z}_{t+s+1}|\widetilde{z}_{t+1}]|z_t] = V[\rho^s \widetilde{z}_{t+1}|z_t] = \rho^{2s} V[\rho z_t + \widetilde{\varepsilon}_{t+1}|z_t] = \rho^{2s}\sigma_\varepsilon^2.$$

Thus the second-order consumption function becomes

$$\begin{aligned} & c_t(x_t, z_t) \\ = \; & \frac{\phi^{T-t}}{(T-t+1)_\phi}\left[x_t + \sum_{i=t+1}^{T} \frac{\mu_i + \rho^{i-t} z_t}{R^{i-t}}\right] \\ & -\frac{\gamma+1}{2} \sum_{i=t+1}^{T} \frac{\phi^{T-t}}{(T-i+1)_\phi} E_t\left[V_{i-1}\left[\sum_{j=i}^{T} \frac{\rho^{j-i}\widetilde{z}_i}{R^{j-t}}\right]\right] \frac{1}{x_t} \\ & +O\left(\frac{1}{x_t^2}\right). \end{aligned}$$

$$
\begin{aligned}
& c_t(x_t, z_t) \\
= \quad & \frac{\phi^{T-t}}{(T-t+1)_\phi}\left[x_t + \sum_{i=t+1}^{T} \frac{\mu_i + \rho^{i-t} z_t}{R^{i-t}}\right] \\
& -\frac{\gamma+1}{2} \sum_{i=t+1}^{T} \frac{\phi^{T-t}}{(T-i+1)_\phi} E_t\left[V_{i-1}\left[\frac{\widetilde{z}_i}{R^{i-t}} \sum_{j=i}^{T} \frac{\rho^{j-i}}{R^{j-i}}\right]\right] \frac{1}{x_t} \\
& +O\left(\frac{1}{x_t^2}\right).
\end{aligned}
$$

$$
\begin{aligned}
& c_t(x_t, z_t) \\
= \quad & \frac{\phi^{T-t}}{(T-t+1)_\phi}\left[x_t + \sum_{i=t+1}^{T} \frac{\mu_i + \rho^{i-t} z_t}{R^{i-t}}\right] \\
& -\frac{\gamma+1}{2} \sum_{i=t+1}^{T} \frac{\phi^{T-t}}{(T-i+1)_\phi} E_t\left[V_{i-1}\left[\frac{\widetilde{z}_i}{R^{i-t}} \sum_{j=0}^{T-i} \left(\frac{\rho}{R}\right)^j\right]\right] \frac{1}{x_t} \\
& +O\left(\frac{1}{x_t^2}\right).
\end{aligned}
$$

$$
\begin{aligned}
& c_t(x_t, z_t) \\
= \quad & \frac{\phi^{T-t}}{(T-t+1)_\phi}\left[x_t + \sum_{i=t+1}^{T} \frac{\mu_i + \rho^{i-t} z_t}{R^{i-t}}\right] \\
& -\frac{\gamma+1}{2} \sum_{i=t+1}^{T} \frac{\phi^{T-t}}{(T-i+1)_\phi} E_t\left[V_{i-1}\left[\frac{\widetilde{z}_i}{R^{i-t}} (T-i+1)_{\frac{\rho}{R}}\right]\right] \frac{1}{x_t} \\
& +O\left(\frac{1}{x_t^2}\right).
\end{aligned}
$$

$$
\begin{aligned}
& c_t(x_t, z_t) \\
= \quad & \frac{\phi^{T-t}}{(T-t+1)_\phi}\left[x_t + \sum_{i=t+1}^{T} \frac{\mu_i + \rho^{i-t} z_t}{R^{i-t}}\right] \\
& -\frac{\gamma+1}{2} \sum_{i=t+1}^{T} \frac{\phi^{T-t} \rho^{2(i-t)} \sigma_\varepsilon^2}{(T-i+1)_\phi} \frac{(T-i+1)_{\frac{\rho}{R}}^2}{R^{2(i-t)}} \frac{1}{x_t} \\
& +O\left(\frac{1}{x_t^2}\right).
\end{aligned}
$$

Note that if we set

$$
\lambda = \frac{1-\rho}{2}
$$

and

$$\sigma_\varepsilon = (1-\rho^2)\sigma^2 = (1-\rho)(1+\rho)\sigma^2 = 2\lambda(2-2\lambda)\sigma^2 = 4\lambda(1-\lambda)\sigma^2,$$

we would get the same result if z_t followed the Markov process from 2.3 but with $y_s(z_s) = \mu_s + \sigma z_s$ for all $s = t+1, ..., T$. With respect to the second-order moments, an AR(1) and a two-state Markov process cannot be distinguished.

3 Solving the Model Numerically

In "Information Shocks and Precautionary Saving", we compare the approximate consumption function of Eq. (25) to a consumption function computed via numerical dynamic programming. For the Markov income process described in the paper, there are n states in each period, and the consumption function for each state and period was approximated by Schumaker shape-preserving splines as described in Judd (1999). Assuming the lowest income state in each period corresponds to the state value Z_1, the minimum possible cash on hand at period t is (Aiyagari (1994))

$$x_t^{\min} = -\sum_{s=t+1}^{T} \frac{y_s(Z_1)}{R^{s-t}}.$$

A suitably large cash on hand X was chosen such that for $x_t > X$, the consumption function was assumed to obey

$$\widehat{c}_t^{(0)}(x_t, z_t) = \frac{\phi^{T-t}}{(T-t+1)_\phi}\left[x_t + \sum_{i=t+1}^{T} \frac{E_t[y_i(\widetilde{z}_i)]}{R^{i-t}}\right]. \tag{54}$$

Between $x_t^{\min}$ and X, the state space was divided into 1000 pieces of geometrically varying length such that the $(i+1)$th piece was 1.01 times as large as the ith piece, which allows for the fact that the consumption function gets more nonlinear as $x \to x_t^{\min}$ and, therefore, the pieces need to be smaller in the vicinity of $x_t^{\min}$.

Given the consumption functions $c_{t+1,i}(x) = c_t(x, Z_i)$, $c_{ti}(x)$ was obtained at the boundaries of the pieces by solving the Euler equation (9). Computing the Schumaker spline also requires $c'_{ti}(x)$. Define

$$D_t(x_t, z_t) = E_t\left[\frac{\partial c_{t+1}}{\partial x_{t+1}}(x_{t+1}(x_t, z_t, \widetilde{z}_{t+1}), \widetilde{z}_{t+1}) c_{t+1}(x_{t+1}(x_t, z_t, \widetilde{z}_{t+1}), \widetilde{z}_{t+1})^{-\gamma-1}\right]$$

Then, differentiating (9) gives

$$\frac{\partial c_t}{\partial x_t}(x_t, z_t) = \frac{\beta R^2 c_t(x_t, z_t)^{\gamma+1} D_t(x_t, z_t)}{1 + \beta R^2 c_t(x_t, z_t)^{\gamma+1} D_t(x_t, z_t)},$$

where the righthand side is known after we have computed $c_{ti}(x)$. At the minimum value, the behavior of the consumption function can be derived analytically (Carroll (2002)):

$$\begin{aligned}
c_{ti}(x_t^{\min}) &= 0 \\
\frac{dc_{ti}}{dx}(x_t^{\min}) &= \left\{1 + \phi^{-1}\pi_{i1}^{1/\gamma}\left[\frac{dc_{t+1,1}}{dx}(x_t^{\min})\right]^{-1}\right\}^{-1} \qquad 0 \le t < T_W \\
\frac{dc_{ti}}{dx}(x_t^{\min}) &= \frac{\phi^{T-t}}{(T-t+1)_\phi} \quad T_W \le t \le T.
\end{aligned}$$

Here, we assume that $\pi_{i1} > 0$ and $y_t(Z_1) < y_t(Z_2)$ for $0 \le t < T_W$ so there is always a positive probability of realizing the minimum income shock and, indeed, π_{i1} is the probability of getting income $y_t(Z_1)$.

For all calibrations of the model, aggregate variables were computed using one million simulations per cohort.

4 Deriving the Variance and Wealth Dynamics for the Simple Lifecycle Model

Let us assume that income follows a Markov process up to some $T_W \le T$, and thereafter income is constant. Formally, y_t is independent of z_t for $T_W < t \le T$ while, for $0 \le t \le T_W$, income is given by $y_t(z_t)$, where z_t is a first-order Markov process that takes on the discrete values $Z_1, \ldots, Z_n$ with transition matrix

$$\pi_{ij} = \Pr[\tilde{z}_{t+1} = Z_j | z_t = Z_i].$$

Given an estimate of the consumption function $\widehat{c}_t(x_t, z_t)$ we can estimate$X_{ti} = E[x_t | z_t = Z_i]$ as follows. First,

$$X_{0i} = y_0(Z_i) \quad i = 1, \ldots, n. \tag{55}$$

Then given $X_{t1}, \ldots, X_{tn}$, the average cash on hands for each state next period are determined by the difference equations

$$X_{t+1,i} = \sum_{j=1}^{n} \pi_{ij}\left[y_{t+1}(Z_i) + R\left(X_{tj} - \widehat{c}_t(X_{tj}, Z_j)\right)\right] \quad i = 1, \ldots, n. \tag{56}$$

We then obtain aggregate variables at age t by averaging over the n states at t with the unconditional distribution p_t:

$$X_t = \sum_{i=1}^{n} p_{ti} X_{ti}. \tag{57}$$

To zeroth-order, the approximate consumption function of (54) can be written

$$\widehat{c}_t^{(0)}(x_t, z_t) = \frac{\phi^{T-t}}{(T-t+1)_\phi}\left[x_t + \sum_{i=t+1}^{T} \frac{E_t[y_i(\widetilde{z}_i)]}{R^{i-t}}\right]. \tag{58}$$

We then define the zeroth-order mean wealth as

$$W_t^{(0)} = X_t^{(0)} + \sum_{i=t+1}^{T} \frac{E[y_i(\widetilde{z}_i)]}{R^{i-t}}, \tag{59}$$

where $X_t^{(0)}$ is defined by (57) using (58) in (56).

Let us define

$$h_t(z_t) = \sum_{j=t}^{T} \frac{E[y_j(\widetilde{z}_j)|z_t]}{R^{j-t}}. \tag{60}$$

This satisfies the expectational difference equation

$$\begin{aligned} h_t(z_t) &= y_t(z_t) + \frac{1}{R}E\left[\sum_{j=t+1}^{T} \frac{E[y_j(\widetilde{z}_j)|\widetilde{z}_{t+1}]}{R^{j-t-1}}\middle| z_t\right] \\ &= y_t(z_t) + \frac{1}{R}E\left[h_{t+1}(\widetilde{z}_{t+1})|z_t\right] \end{aligned} \tag{61}$$

Note that

$$h_{T_W}(z_{T_W}) = Y_{z_{T_W}}.$$

Let us assume that

$$\begin{aligned} h_t(Z_1) &= M_t - S_t \\ h_t(Z_2) &= M_t + S_t, \end{aligned} \tag{62}$$

where M_t and S_t are constants to be determined. Then $M_{T_W} = 1$ and $S_{T_W} = \sigma$. Given M_{t+1} and S_{t+1}, we get

$$\begin{aligned} h_t(Z_1) &= 1 - \sigma + \frac{1+\rho}{2}\frac{M_{t+1} - S_{t+1}}{R} + \frac{1-\rho}{2}\frac{M_{t+1} + S_{t+1}}{R} \\ &= 1 - \sigma + \frac{M_{t+1} - \rho S_{t+1}}{R} \end{aligned}$$

and

$$\begin{aligned} h_t(Z_2) &= 1 + \sigma + \frac{1+\rho}{2}\frac{M_{t+1} + S_{t+1}}{R} + \frac{1-\rho}{2}\frac{M_{t+1} - S_{t+1}}{R} \\ &= 1 + \sigma + \frac{M_{t+1} + \rho S_{t+1}}{R}. \end{aligned}$$

Thus,

$$M_t = 1 + \frac{M_{t+1}}{R} \tag{63}$$

$$S_t = \sigma + \frac{\rho}{R} S_{t+1}, \tag{64}$$

where these difference equations have the solutions

$$M_t = (T_W - t + 1)_{\frac{1}{R}} \tag{65}$$

$$S_t = \sigma (T_W - t + 1)_{\frac{\rho}{R}}. \tag{66}$$

We then have

$$\begin{aligned} E\left[h_{t+1}(\widetilde{z}_{t+1}) | z_t = Z_1\right] &= \frac{1+\rho}{2}\left(M_{t+1} - S_{t+1}\right) + \frac{1-\rho}{2}(M_{t+1} + S_{t+1}) \\ &= M_{t+1} - \rho S_{t+1} \end{aligned}$$

$$\begin{aligned} E\left[h_{t+1}(\widetilde{z}_{t+1}) | z_t = Z_2\right] &= \frac{1-\rho}{2}\left(M_{t+1} - S_{t+1}\right) + \frac{1+\rho}{2}(M_{t+1} + S_{t+1}) \\ &= M_{t+1} + \rho S_{t+1} \end{aligned}$$

Thus,

$$\begin{aligned} & V\left[h_{t+1}(\widetilde{z}_{t+1}) | z_t = Z_1\right] \\ = & \frac{1+\rho}{2}\left(M_{t+1} - S_{t+1} - M_{t+1} + \rho S_{t+1}\right)^2 \\ & + \frac{1-\rho}{2}(M_{t+1} + S_{t+1} - M_{t+1} + \rho S_{t+1})^2 \\ = & \frac{1+\rho}{2}(1-\rho)^2 S_{t+1}^2 + \frac{1-\rho}{2}(1+\rho)^2 S_{t+1}^2 = (1-\rho^2) S_{t+1}^2. \end{aligned}$$

Likewise,

$$\begin{aligned} & V\left[h_{t+1}(\widetilde{z}_{t+1}) | z_t = Z_2\right] \\ = & \frac{1-\rho}{2}\left(M_{t+1} - S_{t+1} - M_{t+1} - \rho S_{t+1}\right)^2 \\ & + \frac{1+\rho}{2}(M_{t+1} + S_{t+1} - M_{t+1} - \rho S_{t+1})^2 \\ = & (1-\rho^2) S_{t+1}^2. \end{aligned}$$

Therefore,

$$V_t[h_{t+1}(\widetilde{z}_{t+1})] = (1-\rho^2) S_{t+1}^2 = (1-\rho^2)\sigma^2 (T_W - t)^2_{\frac{\rho}{R}}.$$

For the wealth dynamics, since $E[y_t(\widetilde{z}_t)] = 1$, (59) gives

$$W_t^{(0)} = X_t^{(0)} + \frac{1}{R}(T_W - t)_{\frac{1}{R}} \tag{67}$$

Since $b_0 = 0$,

$$X_0^{(0)} = 1. \tag{68}$$

From (11) and (54), the dynamics of $X_t^{(0)}$ are given by

$$\begin{aligned} X_{t+1}^{(0)} &= E[y_{t+1}(\widetilde{z}_{t+1})] + R(X_t^{(0)} - E[\widehat{c}_t^{(0)}(X_t^{(0)}, \widetilde{z}_t)]) \\ &= 1 + R\left(X_t^{(0)} - \frac{\phi^{T-t}}{(T-t+1)_\phi}\left[X_t^{(0)} + \sum_{i=t+1}^{T} \frac{E[y_i(\widetilde{z}_i)]}{R^{i-t}}\right]\right) \\ &= 1 + \frac{R}{(T-t+1)_\phi}\left((T-t)_\phi X_t^{(0)} - \frac{\phi^{T-t}}{R}(T_W - t)_{\frac{1}{R}}\right). \end{aligned} \tag{69}$$

Using (67), we obtain

$$\begin{aligned} & W_{t+1}^{(0)} - \frac{1}{R}(T_W - t - 1)_{\frac{1}{R}} \\ = \quad & 1 + \frac{R}{(T-t+1)_\phi}\left((T-t)_\phi W_t^{(0)} - \frac{(T-t)_\phi + \phi^{T-t}}{R}(T_W - t)_{\frac{1}{R}}\right) \end{aligned}$$

Simplifying,

$$W_{t+1}^{(0)} = \frac{R(T-t)_\phi}{(T-t+1)_\phi} W_t^{(0)}. \tag{70}$$

This has the solution

$$W_t^{(0)} = R^t \frac{(T-t+1)_\phi}{(T+1)_\phi} W_0^{(0)}.$$

From (67) and (68),

$$W_0^{(0)} = (T_W + 1)_{\frac{1}{R}},$$

so expected wealth is given by

$$W_t^{(0)} = R^t \frac{(T-t+1)_\phi}{(T+1)_\phi}(T_W + 1)_{\frac{1}{R}}.$$

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