Precautionary Social Planning*

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Abstract

It is a truism of neoclassical economics that a sufficiently high savings rate will be bad if it is dynamically inefficient. While this argument may be compelling in a decentralized framework, it may not be for a social planner. Here we consider a Solow model in which households follow a savings rate that a social planner targets but does not have complete control over. If the third derivative of the production function is sufficiently large at the Golden Rule capital stock, expected utility will be maximized if he targets a savings rate higher than the Golden Rule savings rate. Under these circumstances, dynamic inefficiency will yield greater stability of consumption, which is often a stated priority of social planners. For a Cobb-Douglas production function, this occurs if the Golden Rule savings rate, i.e. the share of capital, is less than 1/2. For a CES production function, the range of Golden Rule savings rates for which dynamic inefficiency is optimal increases as capital and labor become more complementary. China's aggregate savings rate of 40% could be optimal if the elasticity of substitution between capital and labor is between 0.7 and 0.8.

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In recent years, there has been a growing clamor of criticism that the overwhelming majority of macroeconomic models are inhabited by a counterfactual species, often called *Homo economicus*, of perfectly rational beings. The profession has largely clung to this assumption in spite of a dearth of empirical evidence that people actually behave this way and a profusion of theoretical evidence that the benefits to behaving this way are negligible if they are even

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positive.[1] However, there is another ubiquitous assumption in economics that is just as specious but gets little condemnation, at least in Western economics journals. The pure market economies of our models exist nowhere in reality. We do have central authorities that try to coordinate behavior across the disparate participants in our economies, and since the end of the Cold War the predominant trend around the world has been toward more government control rather than away from it.

Consider, for example, the case of China, which, depending on how one compares GDP across countries, is now either the largest or second largest economy on the planet, and within a few years there will be no ambiguity. A large literature models China as a country of decentralized rational households who each individually maximize their utility subject to budget constraints consistent with market equilibria,[2] but does it really make sense to model China that way? In liberal democracies, it is quite natural to assume that households behave with virtual autonomy since that is the fundamental nature of a liberal democracy. China, in contrast, impugns the social instability that it claims results from giving individuals too much freedom.

The aggregate savings rate for China averaged to 40% between 2002 and 2020.[3] This is a much higher rate than what we typically observe in liberal democracies, and it is very difficult to explain with a decentralized model. For example, Song et al (2011) have to assume households are extremely patient with a discount rate of 0.3%, which is an order of magnitude smaller than what we usually assume for Western economies, when constructing a decentralized growth model consistent with Chinese macroeconomic data.

Another framework we could use to study China, one that requires fewer assumptions, is the Solow (1956) model with an endogenous savings rate. It seems much more plausible to suppose that China's leaders are coordinating household behavior to generate high savings than it is to suppose that Chinese households are independently choosing to save more than their Western counterparts. If a social planner chooses a target savings rate to maximize its objectives–which is literally what China purports to do with its five-year plans[4]–it is actually quite easy to understand why China would save at a rate

[1] For examples where rationality actually decreases utility in general equilibrium see Feigenbaum and Caliendo (2010); Feigenbaum, Caliendo, and Gahramanov (2011); and Feigenbaum, Gahramanov, and Tang (2013).

[2] See, for example, Chen and Wen (2017), Garriga et al. (2021), He et al. (2019), and Song et al. (2011).

[3] From the (Chinese) National Bureau of Statistics. For aggregate savings, https://data.stats.gov.cn/easyquery.htm?cn=C01&zb=A020F0102&sj=2020,
and for GNP
https://data.stats.gov.cn/easyquery.htm?cn=C01&zb=A0201&sj=2021.

[4] Five-year plans are passed by the National People's Congress and are tailored to the specific geopolitical circumstances at the time of writing. Most plans specify a target for GDP or GNP growth. For example, the 13th Plan advised that GDP growth should average 6.5% per year from 2016-2020. (The 14th Plan for 2021-2025 is less explicit about short-term goals because of uncertainty resulting from COVID-19.) Since the ability to adjust labor and technology over a five-year horizon is quite limited, the primary instrument for achieving growth targets will be the savings rate. How savings is allocated across sectors will also

close to the Golden Rule savings rate that maximizes steady-state consumption per capita. Deviations of the target savings rate from the Golden Rule savings rate can also be explained. If the social welfare function is expected utility of consumption per capita and the actual savings rate is subject to noise, the social planner will have a precautionary motive to save more or less than the Golden Rule, depending on the parameters of the production function.[5]

Some critics will argue that the Solow model is too simple for use by macroeconomists today. It would be more accurate to say that the Solow model is too simple to address many of the questions that Western economists have about how macroeconomic issues impact voters. The Cass-Koopmans-Ramsey model,[6] overlapping-generations models,[7] and their various successors did not supplant the Solow model because of their empirical successes. All of these models are about equally good–or bad–at macroeconomic forecasting. Since we are asking questions about decision-making at the societal level rather than at the individual level, we do not require the sophistication of these more popular models, and there is no reason to hobble ourselves with their problematic assumptions of perfect rationality and a pure market economy. All we need is a production function and a process for determining the savings rate. We do not even have to distinguish between public and private goods.

The predominant question for most macroecononomists trying to understand China is whether it is saving *too much*. For those of us who are fond of the low-saving liberal democracies that we live in, there is a self-serving interest that the answer to this question be yes. Recent five-year plans have put more emphasis on encouraging consumption, suggesting that China's leaders are also worried that the country has been saving too much. Notwithstanding these sentiments, we find in our simpler framework that the answer is probably no.[8] There is even a possibility that China could achieve its societal goals more expeditiously by saving more.

While there are also Keynesian concerns about Chinese oversaving that we will not consider here, the neoclassical argument is that China's economy may be dynamically inefficient. However, the concept of Pareto efficiency is only relevant in a context where the social planner, whether putative or real, has absolute control over the allocation of goods. To paraphrase Friedman (1953),[9] we need hardly belabor the point that to date there is no reason for confidence in the ability of governments to redistribute goods as desired. If a supposedly Pareto-improving reallocation is not actually feasible, its purported existence is irrelevant.

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matter, but that is beyond the scope of this analysis.

[5] This is similar to Cukierman's (2002) explanation for why central banks target a positive inflation rate. Even though 0% inflation may be optimal, the cost of 2% deflation is much higher than the cost of 2% inflation.

[6] See Cass (1965), Koopmans (1965), and Ramsey (1928).

[7] See Diamond (1965).

[8] This policy advice comes with the caveat that we have, thus far, only considered what happens in the steady state. Optimal policy for a social planner with imperfect control during the transition to the steady state remains unexplored.

[9] p. 129.

If a social planner has absolute control over the savings rate and only cares about consumption in the steady state, as a social planner ought to according to Ramsey (1928), then he would trivially choose the Golden Rule savings rate that maximizes steady-state consumption. However, things are more complicated when the actual savings rate that gets implemented by households is subject to noise. As in Leland (1968) and Sandmo (1970), the optimal choice will depend on the third derivative of the social planner's objective function. Given reasonable assumptions about the utility function, for small noise variances this third derivative does not depend on the choice of the utility function. Only the third derivative of steady-state consumption as a function of the savings rate, evaluated at the Golden Rule savings rate, matters. The sign of this third derivative is particularly noteworthy. If the third derivative is positive, consumption will decline more from its maximum if households save too little than if they save too much. Consequently, it will be optimal for the social planner to target a savings rate that, at least ostensibly, is dynamically inefficient. The third derivative of steady-state consumption is in turn strictly a function of the third and lower derivatives of the production function at the Golden Rule steady state. In general the condition for dynamic inefficiency relates the elasticity corresponding to this third derivative to the elasticity of substitution between capital and labor.

For the case of a Cobb-Douglas production function, the Golden Rule savings rate is the (constant) share of capital, and the third derivative of consumption will be positive if the share of capital is less than one half. If we generalize to a constant elasticity of substitution (CES) production function, the Golden Rule savings rate will depend on both the parameters of the production function and the growth (or decay) rates of the factors of production. The range of Golden Rule savings rates for which dynamic efficiency is optimal depends on the elasticity of substitution between capital and labor. The more complementary capital and labor are, the larger this range will be. This follows because, if the elasticity of substitution is less than one and households do not save enough to maintain enough capital to produce, consumption will converge to zero. The social planner will very much want to avoid this outcome, so the precautionary motive to save more will become especially strong as the elasticity of substitution decreases. If the elasticity is less than 0.5, dynamic inefficiency is always optimal. In contrast, if capital and labor are highly substitutable, capital is not essential for production so a low savings rate is less terrible. In the limit where capital and labor are nearly perfect substitutes, dynamic inefficiency will only be optimal if the Golden Rule savings rate is less than 20%.

For the motivating case of China, the aggregate savings rate has averaged 40% in recent years and the income share of capital has averaged 0.5.[10] Taken at face value, these numbers suggest that China ought to be saving in the vicinity of 50%. However, what matters is the share of capital at the steady state, not the share of capital during the transition. If, as China builds up its capital

[10]The average income share of capital is from 2002 and 2017 according to the *China Statistical Yearbook*.

stock, the share of capital drops to the range of 0.3-0.4 like for most developed countries today, a savings rate of 40% could be optimal. In a model calibrated to match current Chinese macroeconomic data, we find this can happen if the elasticity of substitution between capital and labor is roughly between 0.7 and 0.8.

The paper is organized as follows. In Section 1 we describe the model. In Section 2 we review the Golden Rule, which will give the optimal savings rate if the social planner has perfect control over households. In Section 3 we generalize to the case where the social planner has imperfect control. In Sections 4 and 5 we discuss what happens for the particular cases of a Cobb-Douglas and CES production function respectively. In Section 6 we calibrate the model so China's current savings rate is optimal. We conclude in Section 7.

1 The Model

Our object of study is a closed economy with a constant returns to scale aggregate production function

$$Y(t) = F(K(t), A(t)L(t)) \tag{1}$$

of the factors of production capital $K(t)$, labor $L(t)$, and labor-augmenting technology $A(t)$. We assume that labor and technology grow exogenously at the rates n and g respectively and capital depreciates exogenously at the rate δ. If we define capital per effective labor

$$k(t) = \frac{K(t)}{A(t)L(t)}, \tag{2}$$

and the intrinsic production function

$$f(k) = F(k, 1), \tag{3}$$

we can rewrite the production function as

$$Y(t) = A(t)L(t)f(k(t)).$$

We also assume the function $f(k)$ is positive, strictly increasing, and strictly concave; has a continuous third derivative for $k > 0$; and is nonnegative for $k = 0$. It is helpful to define

$$\xi = n + g + \delta, \tag{4}$$

which we assume is positive. This is the unit cost of maintaining a constant capital per effective labor. We also assume that

$$\lim_{k\to\infty} f'(k) < \xi < \lim_{k\to 0} f'(k). \tag{5}$$

Most of the other assumptions of a macro model relate to how final goods will be divvied up between various end users. Here we only distinguish between whether a final good is installed as capital or consumed so we can dispense with those assumptions. We have

$$Y(t) = C(t) + I(t), \tag{6}$$

where $C(t)$ includes both public and private consumption, and $I(t)$ includes both public and private investment. Investment adds to the capital stock, yielding the equation of motion

$$\frac{dK(t)}{dt} = I(t) - \delta K(t). \tag{7}$$

The main innovation of the paper is to assume that the savings rate $s \in (0, 1)$ such that

$$I(t) = sY(t) \tag{8}$$

is determined by a social planner so as to maximize a social welfare function. We can then differentiate $k(t)$ to obtain the familiar Solow equation of motion

$$\frac{dk(t)}{dt} = sf(k(t)) - \xi k(t). \tag{9}$$

Suppose that s is large enough that there exists $\underline{k} \geq 0$ such that $sf(\underline{k}) > \xi \underline{k}$.[11] Then (5) implies that a $k^*(s) > \underline{k}$ will exist such that

$$sf(k^*(s)) = \xi k^*(s), \tag{10}$$

and the strict concavity of f implies that $k^*(s)$ must be unique.[12] For any $k(0) \geq \underline{k}$, the resulting solution to (9) will satisfy $\lim_{t\to\infty} k(t) = k^*(s)$. Given our assumptions about the smoothness of s, $k^*(s)$ must be continuous and thrice differentiable. Let us then define

$$c^*(s) = (1 - s)f(k^*(s)) \tag{11}$$

as steady state consumption per effective labor, which is also continuous and thrice differentiable.

While the original Solow (1956) paper treated the savings rate s as exogenous, here we endogenize s. The social planner chooses its target savings rate $\widetilde{s}$ to maximize

$$V(\widetilde{s}) = E[u(c^*(\widetilde{s} + \varepsilon))], \tag{12}$$

[11] If $sf(k) < \xi k$ for all $k > 0$, we define $k^*(s) = 0$ since that is the only steady state. Thus $k^*(0) = 0$.

[12] The average productivity of capital $f(k)/k$ must be the same for any steady state with positive k. Since the strict concavity of f implies average productivity is strictly decreasing in k, any such steady state must be unique.

where $u(c)$ is a strictly concave, strictly increasing function with a continuous third derivative for $c > 0$; the actual savings rate is $s = \widetilde{s} + \varepsilon$; and ε is a mean-zero noise variable.[13] Note that this differs substantially from the common utilitarian social welfare function which is a weighted sum of the utilities of individual households. It is, however, consistent with what many socialists advocate. In his "Critique of the Gotha Program", Marx (1875) famously summarized the long-run goal of communism to be that consumption should be allocated such that "From each according to his ability, to each according to his needs!" Since it would be impossible for a social planner to anticipate all of the future needs of households, the objective should presumably be to maximize the size of the pie available to accommodate those needs, i.e. to maximize the available aggregate consumption, which c^* is proportional to. By focusing on the steady state, we also effectively weight all generations the same, avoiding the ethical dilemma of having to play favorites between generations.[14] On the other side, this choice of social welfare function also means the social planner does not care a whit about the Pareto criterion for ranking allocations. Only in the special case where all households have exactly the same preferences as the social planner should we expect equilibrium allocations to be Pareto efficient.[15]

The noise ε in the savings rate arises because the social planner does not have full control in coordinating the behavior of households. Households will almost surely have an incentive to deviate from the behavior prescribed by the social planner so the variance of ε will increase as households are more willing to act independently. But even if households are entirely docile, coordination failures may still occur.

To connect what we are doing with a more traditional macro model, let us suppose that we have M households. The ith household earns labor income y_t^i at t and has saving b_t^i that earns the interest rate r_t. Then we normally assume the household will choose its consumption c_t^i and saving for next period b_{t+1}^i to maximize a utility function subject to the budget constraint

$$c_t^i + b_{t+1}^i = y_t^i + (1 + r_t)b_t^i. \qquad (13)$$

Then next period's capital stock is determined by aggregating the savings

$$K_{t+1} = \sum_{i=1}^{M} b_{t+1}^i. \qquad (14)$$

What we are doing instead is to assume instead that there is some $s \in [0, 1]$

[13] Assuming $E[\varepsilon] = 0$ is tantamount to assuming the social planner has rational expectations since he can offset any bias in ε by adjusting s^* accordingly.

[14] While it is commonly believed that the infinity of the social welfare function at the Golden Rule makes it impossible to calculate optimal transitions to the Golden Rule steady state, this turns out not to be true. We can, like quantum field theorists, simply ignore the infinity of the objective function when computing the optimal path of the choice variables. However, we leave the study of the optimal transition to future work.

[15] It is perhaps worth noting that in a market economy with distortionary taxes we should not expect equilibrium allocations to be Pareto efficient either.

such that the average labor income

$$\overline{y}_t = \frac{1}{M}\sum_{i=1}^{M} y_t^i \tag{15}$$

and average savings

$$\overline{b}_t = \frac{1}{M}\sum_{i=1}^{M} b_t^i \tag{16}$$

satisfy

$$\overline{b}_{t+1} = s(\overline{y}_t + r_t\overline{b}_t) + (1-\delta)\overline{b}_t, \tag{17}$$

and again next period's capital stock is determined by (14) and equals $M\overline{b}_{t+1}$. However the b_{t+1}^i are each chosen, we could always solve (17) for s. The only actual content to this assumption is that s remains constant in time.[16] For an economy where there is a central authority, it seems more sensible to assume the planner will try to induce households to follow

$$b_{t+1}^i = \widetilde{s}(y_t^i + r_t b_t^i) + (1-\delta)b_t^i \tag{18}$$

than to assume each household maximizes its utility without any coordination.

2 The Golden Rule

We begin by reviewing the familiar textbook example of the model without uncertainty. In that case, the social planner's objective function reduces to $V(\widetilde{s}) = U(\widetilde{s})$, where

$$U(s) = u(c^*(s)). \tag{19}$$

The first-order condition for the social planner's problem is

$$U'(s) = u'(c^*(s))\frac{dc^*(s)}{ds} = 0. \tag{20}$$

Since u is strictly increasing, the social planner needs to find a solution to $\frac{dc^*(s)}{ds} = 0$, what is commonly known as the Golden Rule savings rate.

The steady state is defined by (10), so we can rewrite steady state consumption as a function of k^* alone:

$$c^*(s) = f(k^*(s)) - \xi k^*(s). \tag{21}$$

Thus

$$\frac{dc^*(s)}{ds} = [f'(k^*(s)) - \xi]\frac{dk^*(s)}{ds}. \tag{22}$$

[16] Even that assumption is made without loss of generality here since for this paper we only consider what happens in the steady state.

Implicit differentiation of (10) gives us

$$\frac{dk^*(s)}{ds} = \frac{f(k^*(s))}{\xi - sf'(k^*(s))}. \tag{23}$$

However, we have as an identity that

$$\xi = \frac{sf(k^*(s))}{k^*(s)} \tag{24}$$

for all s, so we can rewrite (23) as

$$\frac{dk^*(s)}{ds} = \frac{1}{\frac{f(k^*(s))}{k^*(s)} - f'(k^*(s))} \frac{f(k^*(s))}{s}. \tag{25}$$

Finally, let us define the share of capital as

$$\alpha(k) = \frac{(r(k)+\delta)K}{Y} = \frac{kf'(k)}{f(k)} = \frac{d\ln f(k)}{d\ln k}. \tag{26}$$

Since f is strictly increasing and strictly concave for all k and strictly positive for $k > 0$, we also have that $\alpha(k) \in (0,1)$ for $k > 0$. Using (26) to simplify (25), we obtain

$$\frac{dk^*(s)}{ds} = \frac{1}{1-\alpha(k^*(s))} \frac{k^*(s)}{s}, \tag{27}$$

which is strictly positive if $s, k^*(s) > 0$, in which case the corresponding elasticity works out very simply to

$$\frac{d\ln k^*(s)}{d\ln s} = \frac{1}{1-\alpha(k^*(s))} > 0. \tag{28}$$

Then (22) implies that $\frac{dc^*(s)}{ds} = 0$ iff

$$f'(k^*(s)) = \xi. \tag{29}$$

By (5) and the strict concavity of f, there exists a unique $k_{gr} > 0$ such that $f'(k_{gr}) = \xi$. We have $k^*(0) = 0$ and $k^*(1)$ solves $f(k^*(1)) = \xi k^*(1)$. Since f is strictly concave,

$$f'(k^*(1)) < \frac{f(k^*(1))}{k^*(1)} = \xi = f'(k_{gr}).$$

Therefore $k^*(1) > k_{gr}$. Since $k^*(s)$ is continuous and strictly increasing, and $k_{gr} \in (k^*(0), k^*(1))$, there must exist a unique $s_{gr} \in (0,1)$, which we term the Golden Rule savings rate, that satisfies $k^*(s_{gr}) = s_{gr}$.

We can also employ (24) and (26) to rewrite (29) as

$$\alpha(k^*(s_{gr})) = s_{gr}. \tag{30}$$

Thus the Golden Rule savings rate will in fact equal the share of capital at the Golden Rule steady state.

Note that s_{gr} is indeed a maximum of $U(s)$. The second derivative is

$$U''(s) = u''(c^*(s)) \left(\frac{dc^*(s)}{ds}\right)^2 + u'(c^*(s)) \frac{d^2c^*(s)}{ds^2}. \tag{31}$$

Since u is strictly concave, the first term is unambiguously negative for all s. Differentiating (22) again,

$$\frac{d^2c^*(s)}{ds^2} = f''(k^*(s)) \left(\frac{dk^*(s)}{ds}\right)^2 + [f'(k^*(s)) - \xi] \frac{d^2k^*(s)}{ds^2}, \tag{32}$$

so

$$\frac{d^2c^*(s_{gr})}{ds^2} = f''(k^*(s_{gr})) \left(\frac{dk^*(s_{gr})}{ds}\right)^2 < 0. \tag{33}$$

Thus at the Golden Rule savings rate, the second term of (31) is also negative, so $U''(s_{gr}) < 0$. In the absence of uncertainty, the solution to the social planner's problem is to set the savings rate equal to the share of capital.

The preceding treatment, however, is quite different from how one normally investigates this model. Instead, the focus is usually on which savings rates generate allocations that are Pareto inefficient regardless of preferences. While $U(0)$ and $U(1)$ may not be defined, clearly $c^*(0) = c^*(1) = 0$. Since $c^*(s)$ is continuous, if $s \in (s_{gr}, 1)$, there must exist $s' \in (0, s_{gr})$ such that $c^*(s) = c^*(s')$. However, $k^*(s) > k^*(s')$. Thus if the economy consumes $k^*(s) - k^*(s')$ in an instant, it will immediately be able to resume a stable trajectory with a new steady state capital per effective labor $k^*(s')$ wile maintaining the same steady state consumption per effective labor $c^*(s)$ as before. Since by this transition from the steady state with savings rate s to a savings rate s' it is possible to increase consumption at one instant while maintaining the same consumption at all other instants, an economy with savings rate $s > s_{gr}$ is said to be dynamically inefficient.

What often goes unnoticed in discussing dynamic inefficiency, though, is that the possibility of costless improvement will be purely hypothetical if the economy is not able to precisely adjust its savings rate. In the following, we relax the assumption that the social planner has perfect control over the savings rate.

3 General Results with Imperfect Control

The main innovation of the paper is to suppose that a real-world social planner may not have perfect control over the behavior of his charges. If the social planner can choose a target savings rate $\widetilde{s}$, but the actual savings rate $s = \widetilde{s} + \varepsilon$, where ε is a mean-zero noise variable, then the social planner's objective function will be

$$V(\widetilde{s}) = E[U(\widetilde{s} + \varepsilon)]. \tag{34}$$

We are not interested in a figurehead social planner. The actual savings rate should be reasonably close to the target savings rate, so we assume the variance $V[\varepsilon] = \tau^2$ is small compared to 1.[17]

The social planner will optimally choose $\widetilde{s}$ so that

$$V'(\widetilde{s}) = E[U'(\widetilde{s} + \varepsilon)] = 0. \tag{35}$$

We will treat the dimensionless τ as a perturbation parameter around the optimal choice of $\widetilde{s} = s_{gr}$ when $\tau = 0$. Then the nth moment of ε will be to nth order in τ unless the moment vanishes. Let us define

$$\Delta s = \widetilde{s} - s_{gr}. \tag{36}$$

Since Δs vanishes when $\tau = 0$, Δs must be at least first order in τ. A Taylor expansion of (35) to second order in τ yields

$$E[U'(s_{gr}) + U''(s_{gr})(\Delta s + \varepsilon) + \frac{1}{2}U'''(s_{gr})(\Delta s + \varepsilon)^2] + O(\tau^3) = 0, \tag{37}$$

which simplifies to

$$U''(s_{gr})\Delta s + \frac{1}{2}U'''(s_{gr})(\Delta s^2 + \tau^2) + O(\tau^3) = 0. \tag{38}$$

Note that if Δs is first order in τ, $U''(s_{gr})\Delta s$ would be the only first-order term in (38). Since $U''(s_{gr}) < 0$, any first-order contribution to Δs must vanish. Therefore, we can disregard the Δs^2 term in (38) since it must be fourth order in τ. We are left with

$$U''(s_{gr})\Delta s + \frac{1}{2}U'''(s_{gr})\tau^2 + O(\tau^3) = 0,$$

which gives that the optimal buffer between the target savings rate and the Golden Rule savings rate is

$$\Delta s = -\frac{1}{2}\frac{U'''(s_{gr})}{U''(s_{gr})}\tau^2 + O(\tau^3). \tag{39}$$

The minus sign is cancelled by the $U''(s_{gr})$, so Δs has the same sign as $U'''(s_{gr})$. If $U'''(s_{gr}) > 0$, U will decline more slowly as we move away from s_{gr} to the right than if we move away from s_{gr} to the left. Erring on the side of caution, the social planner should choose $\widetilde{s} > s_{gr}$ with Δs proportional to the variance of ε. Conversely, if $U'''(s_{gr}) < 0$, he should do the opposite.[18]

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[17] Since the social planner will primarily be concerned with avoiding a savings rate that is so high or low that steady-state consumption is zero, we can presume that $\widetilde{s} + \varepsilon < 0$ yields a savings rate of 0 and $\widetilde{s} + \varepsilon > 1$ yields a savings rate of 1.

[18] For more details on optimal precautionary behavior, see Leland (1968) and Sandmo (1970).

Thus everything for the social planner hinges on the third derivative of U at the Golden Rule. Since

$$U'''(s) = u'''(c^*(s)) \left(\frac{dc^*(s)}{ds}\right)^3 + 3u''(c^*(s))\frac{dc^*(s)}{ds}\frac{d^2c^*(s)}{ds^2} + u'(c^*(s))\frac{d^3c^*(s)}{ds^3}, \tag{40}$$

this works out to

$$U'''(s_{gr}) = u'(c^*(s_{gr}))\frac{d^3c^*(s_{gr})}{ds^3}.$$

We have assumed marginal utility is positive, so the sign of the third derivative of U is completely independent of the utility function u. Actually, (31) evaluated at s_{gr} implies that all appearances of the utility function u disappear from (39), so the magnitude of Δs also does not depend on u to second order in τ.

Differentiating (32) once more,

$$\begin{aligned}\frac{d^3c^*(s)}{ds^3} &= f'''(k^*(s))\left(\frac{dk^*(s)}{ds}\right)^3 + 2f''(k^*(s))\frac{dk^*(s)}{ds}\frac{d^2k^*(s)}{ds^2} \\ &\quad + f''(k^*(s))\frac{dk^*(s)}{ds}\frac{d^2k^*(s)}{ds^2} + [f'(k^*(s)) - \xi]\frac{d^3k^*(s)}{ds^3}.\end{aligned} \tag{41}$$

At the Golden Rule savings rate, this reduces to

$$\frac{d^3c^*(s_{gr})}{ds^3} = f'''(k^*(s_{gr}))\left(\frac{dk^*(s_{gr})}{ds}\right)^3 + 3f''(k^*(s_{gr}))\frac{dk^*(s_{gr})}{ds}\frac{d^2k^*(s_{gr})}{ds^2}. \tag{42}$$

This third derivative can be expressed most transparently in terms of elasticities of the production function. We define

$$\alpha_n(k) = \frac{kf^{(n)}(k)}{f^{(n-1)}(k)} \tag{43}$$

to be the elasticity of the nth derivative of the intrinsic production function with respect to capital per effective labor. The derivative of $\alpha_n(k)$ is

$$\begin{aligned}\frac{d\alpha_n(k)}{dk} &= \frac{\left(f^{(n)}(k) + kf^{(n+1)}(k)\right)f^{(n-1)}(k) - kf^{(n)}(k)^2}{f^{(n-1)}(k)^2} \\ &= (1 + \alpha_{n+1}(k) - \alpha_n(k))\frac{\alpha_n(k)}{k}.\end{aligned}$$

So the elasticity of $\alpha_n(k)$ is

$$\frac{kd\alpha_n(k)}{\alpha_n dk} = 1 + \alpha_{n+1}(k) - \alpha_n(k). \tag{44}$$

If we interpret $f^{(0)}(k)$ as the production function, then $\alpha_1(k)$ is the share of capital. The elasticity $\alpha_2(k)$ is usually interpreted as a dimensionless measure of the curvature of the production function. Let $\sigma(k)$ denote the elasticity of substitution between capital and labor, which, from Feigenbaum (2019), can be written in terms of these elasticities as

$$\sigma(k) = -\frac{1-\alpha(k)}{\alpha_2(k)}. \tag{45}$$

The geometric interpretation of $\alpha_3(k)$ is called aberrancy (Schot (1978)). This measures how asymmetric the deviation of the production function is from a quadratic tangent to the production function.

Isolating some unambiguously negative common factors out of (42), we get

$$\begin{aligned}\frac{d^3c^*(s_{gr})}{ds^3} &= \left[\alpha_3(k^*(s_{gr}))\frac{s}{k^*(s_{gr})}\frac{dk^*(s_{gr})}{ds} + 3\frac{s}{\frac{dk^*(s_{gr})}{ds}}\frac{d^2k^*(s_{gr})}{ds^2}\right] \\ &\quad \times \frac{f''(k^*(s_{gr}))}{s}\left(\frac{dk^*(s_{gr})}{ds}\right)^2. \end{aligned} \tag{46}$$

We have already computed the elasticity of k^*, so the final ingredient is the elasticity of $dk^*(s)/ds$.

Differentiating (27),

$$\frac{d^2k^*(s)}{ds^2} = \frac{1}{s}\left[\frac{1-\alpha\left(k^*(s)\right)+k^*(s)\alpha'(k^*(s))}{(1-\alpha(k^*(s)))^2}\right]\frac{dk^*(s)}{ds} - \frac{1}{1-\alpha(k^*(s))}\frac{k^*(s)}{s^2}. \tag{47}$$

We rearrange (44) for $n=1$ to obtain

$$\alpha'(k^*(s)) = \left[1+\alpha_2(k^*(s))-\alpha(k^*(s))\right]\frac{\alpha(k^*(s))}{k^*(s)}. \tag{48}$$

Substituting in this result and (27),

$$\begin{aligned}\frac{d^2k^*(s)}{ds^2} &= -\frac{1}{1-\alpha(k^*(s))}\frac{k^*(s)}{s^2} + \left[\frac{1}{1-\alpha(k^*(s))}\right. \\ &\quad \left. + \frac{\left[1+\alpha_2(k^*(s))-\alpha(k^*(s))\right]\alpha(k^*(s))}{(1-\alpha(k^*(s)))^2}\right]\frac{1}{1-\alpha(k^*(s))}\frac{k^*(s)}{s^2}.\end{aligned}$$

This simplifies to

$$\frac{d^2k^*(s)}{ds^2} = \left[2+\frac{\alpha_2(k^*(s))}{1-\alpha(k^*(s)}\right]\frac{\alpha(k^*(s))}{1-\alpha(k^*(s))}\frac{1}{s}\frac{dk^*(s)}{ds}. \tag{49}$$

Thus the elasticity is

$$\frac{s}{\frac{dk^*(s)}{ds}}\frac{d^2k^*(s)}{ds^2} = \frac{\alpha(k^*(s))}{1-\alpha(k^*(s)}\left[2+\frac{\alpha_2(k^*(s))}{1-\alpha(k^*(s)}\right]. \tag{50}$$

Inserting (28) and (50) into (46), we arrive at our principal result. The third derivative of steady-state consumption per effective labor is

$$\begin{aligned}\frac{d^3c^*(s_{gr})}{ds^3} &= \left[\alpha_3(k^*(s_{gr})) + 3\alpha_2(k^*(s_{gr}))\frac{\alpha(k^*(s_{gr}))}{1-\alpha(k^*(s_{gr}))} + 6\alpha(k^*(s_{gr}))\right] \\ &\quad \times \frac{f''(k^*(s_{gr}))}{s(1-\alpha(k^*(s_{gr}))}\left(\frac{dk^*(s_{gr})}{ds}\right)^2.\end{aligned} \tag{51}$$

Note that the factors outside the brackets are still unambiguously negative. Thus the condition for dynamic inefficiency to be optimal is that

$$\alpha_3(k^*(s_{gr})) + 3\alpha_2(k^*(s_{gr}))\frac{\alpha(k^*(s_{gr}))}{1-\alpha(k^*(s_{gr}))} + 6\alpha(k^*(s_{gr})) < 0. \tag{52}$$

Taking (45) into account, we can further simplify the condition to

$$\alpha_3(k^*(s_{gr})) < 3\alpha(k^*(s_{gr}))\left(\frac{1}{\sigma(k^*(s_{gr}))} - 2\right). \tag{53}$$

Since $\alpha_3(k) = \frac{kf'''(k)}{f''(k)}$, this elasticity related to the aberrancy of the production function will be negative if $f'''(k^*(s_{gr})) > 0$. Thus (53) means that if the third derivative of the production function at the Golden Rule steady state is sufficiently large then dynamic inefficiency will be optimal. Since the share of capital is positive, the right-hand side of (53) is decreasing in the elasticity of substitution. As capital and labor become more substitutable, the bound on $f'''(k^*(s_{gr}))$ such that dynamic inefficiency is optimal gets tighter. Conversely, in the limit as capital and labor are perfect complements, dynamic inefficiency will always be optimal. This is a consequence of capital being essential for production. If the savings rate is sufficiently low, even though it is positive, households will not be able to maintain a positive capital stock, so the social planner will be strongly incentivized to err on the side of higher saving regardless of other parameters of the production function.

From (39), (32) and (41) imply that the optimal ratio of the buffer between the target savings rate and the Golden Rule savings rate to the variance of ε is in the limit as this variance goes to zero

$$\lim_{\tau\to 0}\frac{\Delta s}{\tau^2} = -\frac{1}{2}\frac{\alpha_3(k^*(s_{gr})) + 3\alpha_2(k^*(s_{gr}))\frac{s_{gr}}{1-s_{gr}} + 6s_{gr}}{(1-s_{gr})s_{gr}}. \tag{54}$$

4 Cobb-Douglas Production

The Cobb-Douglas production function

$$f(k) = k^\alpha \tag{55}$$

is the simplest and most commonly used production function precisely because the share of capital $\alpha(k) = \alpha$, which also equals the Golden Rule savings rate, is constant. Indeed, all of the elasticities $\alpha_n = \alpha - n + 1$ are constant, which follows from induction. If $\frac{d\alpha_k(k)}{dk} = 0$, then (44) gives

$$\alpha_{n+1} = \alpha_n - 1 = \alpha - n + 1 - 1 = \alpha - n. \tag{56}$$

The condition (52) then reduces to

$$\alpha - 2 + 3(\alpha - 1)\frac{\alpha}{1-\alpha} + 6\alpha < 0$$

or

$$4\alpha < 2.$$

Dynamic inefficiency will be socially optimal with Cobb-Douglas production as long as the share of capital is less than one half.

The optimal ratio of the buffer to the noise variance is for small τ

$$\lim_{\tau \to 0} \frac{\Delta s}{\tau^2} = -\frac{1}{2}\frac{\alpha - 2 + 3(\alpha-1)\frac{\alpha}{1-\alpha} + 6\alpha}{(1-\alpha)\alpha} = -\frac{1}{2}\frac{\alpha - 2 - 3\alpha + 6\alpha}{(1-\alpha)\alpha} = \frac{1-2\alpha}{(1-\alpha)\alpha}. \tag{57}$$

Since

$$\frac{d}{d\alpha}\left(\frac{1-2\alpha}{(1-\alpha)\alpha}\right) = -\frac{(1-\alpha)^2 + \alpha^2}{(1-\alpha)^2\alpha^2} < 0,$$

the optimal buffer to variance ratio is strictly decreasing in the share of capital. The buffer will be larger in magnitude the farther that the share of capital gets from one half.[19]

5 CES Production

The situation is more complicated for other members of the larger class of CES production functions. For one thing, not all production functions in this class satisfy both inequalities of the assumption (5). The inhabitants of an economy where the production function violates the inequality for $k \to \infty$ can rejoice because for them economics is not a dismal science. Their economy is a generalization of an AK model with $A > \xi$, so a high enough savings rate will permit their consumption per effective labor to grow without bound. In contrast, if the production function violates the inequality for $k \to 0$, no savings rate is high enough to maintain a positive capital stock in the long run.

In this class of production functions,

$$f(k) = [\mu k^\eta + 1 - \mu]^{\frac{1}{\eta}} \tag{58}$$

for $\eta \in (-\infty, 0) \cup (0, 1)$ and $\mu \in (0, 1)$. The Cobb-Douglas case is nested within this family since, as $\eta \to 0$, $f(k) \to k^\mu$. As we will see below, the elasticity of substitution is, consistent with the name of this class, constant and equal to

$$\sigma = \frac{1}{1-\eta} > 0. \tag{59}$$

[19] In general if the optimal buffer vanishes that will generally mean we have to consider contributions to Δs that are higher order in τ. For the special case of a Cobb-Douglas function, however, since $c^*(s) = (1-s)s$ is perfectly symmetric around $s = \frac{1}{2}$ the social planner would presumably be completely indifferent between oversaving and undersaving by the same amount.

When $\eta > 0$, $\sigma > 1$, so we say in this case that capital and labor are substitutes. When $\eta < 0$, $\sigma < 1$, so we say in this case that capital and labor are complements. In the Cobb-Douglas case of $\eta = 0$, capital and labor are neither complements nor substitutes.

One of the advantages of a CES production function is that we get an analytic solution for the steady-state capital per effective labor. If we raise both sides of the steady-state equation

$$s[\mu k^*(s)^\eta + 1 - \mu]^{\frac{1}{\eta}} = \xi k^*(s)$$

to the ηth power, we get an equation linear in $k^*(s)^\eta$. Thus

$$k^*(s) = \left(\frac{1-\mu}{\left(\frac{\xi}{s}\right)^\eta - \mu} \right)^{\frac{1}{\eta}}. \tag{60}$$

However, this solution is only well-defined if

$$\left(\frac{\xi}{s}\right)^\eta > \mu. \tag{61}$$

For $\eta \gtrless 0$, this condition simplifies to $s \lessgtr \frac{\xi}{\mu^{1/\eta}}$. Since

$$f'(k) = \mu \left(\frac{k}{[\mu k^\eta + 1 - \mu]^{\frac{1}{\eta}}} \right)^{\eta - 1} = \mu \left(\frac{k}{f(k)} \right)^{\eta-1}, \tag{62}$$

the significance of $\mu^{1/\eta}$ is that

$$\mu^{1/\eta} = \begin{cases} \lim_{k \to 0} f'(k) & \eta < 0 \\ \lim_{k \to \infty} f'(k) & \eta > 0 \end{cases}. \tag{63}$$

If $\eta > 0$, the assumption (5) implies that $\frac{\xi}{\mu^{1/\eta}} > 1$, so for all $s \in (0,1)$, (61) will be satisfied. If, on the other hand, we have $\frac{\xi}{\mu^{1/\eta}} < 1$, then for sufficiently large s (9) will give $\frac{dk(t)}{dt} > 0$ for all t so there is no steady state.[20]

Conversely, if $\eta < 0$, (5) implies that $\frac{\xi}{\mu^{1/\eta}} < 1$, which means (61) is satisfied for $s \in \left(\frac{\xi}{\mu^{1/\eta}}, 1\right)$. If, on the other hand, we have $\frac{\xi}{\mu^{1/\eta}} > 1$, then, for all $s \in (0,1)$, (9) will give $\frac{dk(t)}{dt} < 0$ if $k(t) > 0$. This follows since $f(0) = 0$ and, since f is strictly concave, $f'(k) < \xi$ for all k, so $sf(k) < f(k) < \xi k$ for all $k > 0$. In this case, however, even if (5) holds and $\frac{\xi}{\mu^{1/\eta}} < 1$, for $s < \frac{\xi}{\mu^{1/\eta}}$, we will still have $\frac{dk(t)}{dt} < 0$ if $k(t) > 0$. This means for $\eta < 0$, the social planner will want to be sure that s does not fall below

$$\underline{s} = \frac{\xi}{\mu^{1/\eta}} \tag{64}$$

[20] This follows since $f(0) = (1-\mu)^{1/\eta} > 0$ and, since f is strictly concave, $f'(k) \geq \xi$ for all k, so $f(k) > \xi k$ for all k.

since $c^*(s) = 0$ for $s < \underline{s}$.

Let us assume in what follows that (5) does hold. Then, applying (62), the share of capital is

$$\alpha(k) = \frac{kf'(k)}{f(k)} = \frac{\mu k^\eta}{\mu k^\eta + 1 - \mu}. \tag{65}$$

Note, as mentioned previously, that for the Cobb-Douglas case of $\eta = 0$ the share of capital simplifies to the constant μ. In that instance, the Golden Rule savings rate that satisfies (30) is obviously μ too.

More generally, we have

$$\frac{d\alpha(k)}{dk} = \frac{\mu\eta k^{\eta-1}(\mu k^\eta + 1 - \mu) - \mu k^\eta(\mu\eta k^{\eta-1})}{(\mu k^\eta + 1 - \mu)^2} = \frac{(1-\mu)\mu\eta k^{\eta-1}}{(\mu k^\eta + 1 - \mu)^2}, \tag{66}$$

so it is not quite so trivial to solve (30) for s_{gr}. Whether the share of capital increases or decreases with k depends on the sign of η. If $\eta > 0$, so capital and labor are substitutes, the share of capital will increase with k. If $\eta < 0$, so capital and labor are complements, the share will decrease with k.

Combining (65) and (60), we have

$$\begin{aligned} s_{gr} &= \frac{\mu\left(\frac{1-\mu}{\left(\frac{\xi}{s_{gr}}\right)^\eta - \mu}\right)}{\mu\left(\frac{1-\mu}{\left(\frac{\xi}{s_{gr}}\right)^\eta - \mu}\right) + 1 - \mu} \\ &= \frac{\frac{\mu}{\left(\frac{\xi}{s_{gr}}\right)^\eta - \mu}}{\frac{\mu}{\left(\frac{\xi}{s_{gr}}\right)^\eta - \mu} + 1} \\ &= \frac{\mu}{\mu + \left(\frac{\xi}{s_{gr}}\right)^\eta - \mu} \\ &= \mu\left(\frac{s_{gr}}{\xi}\right)^\eta. \end{aligned}$$

Thus for CES production, the Golden Rule savings rate is

$$s_{gr} = \left(\frac{\mu}{\xi^\eta}\right)^{\frac{1}{1-\eta}}. \tag{67}$$

Using (59), we can also rewrite this as

$$s_{gr} = \left(\frac{\mu}{\xi^\eta}\right)^\sigma. \tag{68}$$

As we discussed above, the assumption (5) implies that $\frac{\xi}{\mu^{1/\eta}} \gtrless 1$ iff $\eta \gtrless 0$, so $s_{gr} < 1$ since $\sigma > 0$.

The elasticity of the share of capital is, from (66),

$$\frac{k}{\alpha(k)}\frac{d\alpha(k)}{dk} = \eta\frac{1-\mu}{\mu k^\eta + 1 - \mu} = \eta(1 - \alpha(k)). \tag{69}$$

Then (44) gives

$$\alpha_2(k) = \frac{k}{\alpha(k)}\frac{d\alpha(k)}{dk} - (1-\alpha(k)) = (\eta-1)(1-\alpha(k)) < 0. \tag{70}$$

Thus we confirm via (45) that the elasticity of substitution is $\frac{1}{1-\eta}$. Applying (66) again,

$$\frac{d\alpha_2(k)}{dk} = (1-\eta)\frac{d\alpha(k)}{dk} = (1-\eta)\frac{(1-\mu)\mu\eta k^{\eta-1}}{(\mu k^\eta + 1-\mu)^2} = (1-\eta)\eta\frac{\alpha(k)(1-\alpha(k))}{k}. \tag{71}$$

The corresponding elasticity is

$$\frac{k}{\alpha_2(k)}\frac{d\alpha_2(k)}{dk} = \frac{k}{(\eta-1)(1-\alpha(k))}(1-\eta)\eta\frac{\alpha(k)(1-\alpha(k))}{k} = -\eta\alpha(k). \tag{72}$$

Finally, we use (44) once more to obtain the critical elasticity $\alpha_3(k)$:

$$\alpha_3(k) = \frac{k}{\alpha_2(k)}\frac{d\alpha_2(k)}{dk} - (1-\alpha_2(k)) = -\eta\alpha(k) + (\eta-1)(1-\alpha(k)) - 1$$

The aberrancy of a CES production function is

$$\alpha_3(k) = \eta - 2 + (1-2\eta)\alpha(k), \tag{73}$$

which is of ambiguous sign.

For the CES class of production functions, the dynamic inefficiency condition (53) is

$$\eta - 2 + (1-2\eta)s_{gr} - 3s_{gr}(1-\eta-2) < 0, \tag{74}$$

which simplifies to

$$(\eta+4)s_{gr} < 2-\eta. \tag{75}$$

Since $2-\eta$ is strictly positive, we can write this as a lower bound on the inverse Golden Rule savings rate:

$$\frac{1}{s_{gr}} > \frac{\eta+4}{2-\eta}. \tag{76}$$

Since

$$\eta = \frac{\sigma-1}{\sigma},$$

we can also express the lower bound in terms of the elasticity of substitution:

$$\frac{1}{s_{gr}} > \frac{5\sigma-1}{\sigma+1}. \tag{77}$$

For $\sigma \leq 0.2$, the lower bound is negative and obviously satisfied for any possible Golden Rule savings rate. For $\sigma > 0.2$, we can flip (77) to obtain

$$s_{gr} < \frac{\sigma+1}{5\sigma-1}. \tag{78}$$

The right-hand side is a decreasing function of σ for $\sigma > 0.2$. Indeed, the bound only begins to bind for $\sigma > 0.5$. In the limit as capital and labor become perfect substitutes, dynamic inefficiency will still be optimal for $s_{gr} < 0.2$.

Recall that if $\sigma < 1$ (or $\eta < 0$), that $c^*(s) = 0$ for $s \leq \underline{s}$. Using (64) and (67), we have

$$\underline{s} = s_{gr}^{\frac{\eta-1}{\eta}} = s_{gr}^{\frac{1}{1-\sigma}} < s_{gr}. \tag{79}$$

In the limit as $\sigma \to 1$, $\underline{s} \to 0$. However, as σ decreases from 1, $\underline{s}$ gets larger, approaching s_{gr}, which in turn approaches $\xi < 1$, in the limit as capital and labor become perfect complements. When capital and labor become more complementary, capital becomes more essential to production, so the social planner cannot afford to gamble on the side of dynamic efficiency and possibly risk the collapse of the economy.

Since the numerator of (54) is (74), we get that the optimal ratio of the buffer to the noise variance for small τ is

$$\lim_{\tau \to 0} \frac{\Delta s}{\tau^2} = -\frac{1}{2} \frac{\eta - 2 + (4+\eta)s_{gr}}{(1-s_{gr})s_{gr}}. \tag{80}$$

In Fig. 1 we show how steady state consumption varies with the savings rate for various choices of η with $\mu = 0.33$ and $\xi = 0.1$, which are both typical calibrations of μ and ξ for the United States when $\eta = 0$. For the Cobb-Douglas case of $\eta = 0$, steady state consumption is maximized at $s = \mu$. Since $\mu < 0.5$, consumption falls off more quickly for lower savings rates than for higher savings rates, so the social planner would prefer a dynamically inefficient savings rate higher than the Golden Rule. As we make η negative, the Golden Rule savings rate, i.e. the peak of the graph, decreases, and the falloff for $s < s_{gr}$ becomes much steeper. This is in part because $c^*(s)$ is only positive for $s > \underline{s}$ and $\underline{s}$ gets closer to s_{gr} as η decreases. Conversely, if we make η positive, then s_{gr} increases. For $\eta = 0.4$, the Golden Rule savings rate is 0.73. Since s_{gr} is closer to 1 than 0, the falloff is steeper to the right than to the left, so the social planner would prefer a dynamically efficient savings rate less than the Golden Rule.

6 Application to China

In calibrating the model to China, we do not assume that China has already converged to a steady state. Instead, we assume China's future macroeconomic history will be generated by (9) starting from the current capital per effective labor $k(0)$ with a fixed saving rate of $s = 0.433$ that is the average household savings rate between 2002 and 2020. Both Chen and Wen (2017) and Song et al. (2011) use $n = 0.03$ per year, $\delta = 0.1$ per year, and $g = 0.038$ per year. They also both set the income share of capital $\alpha(k(0)) = 0.5$, consistent with recent data.

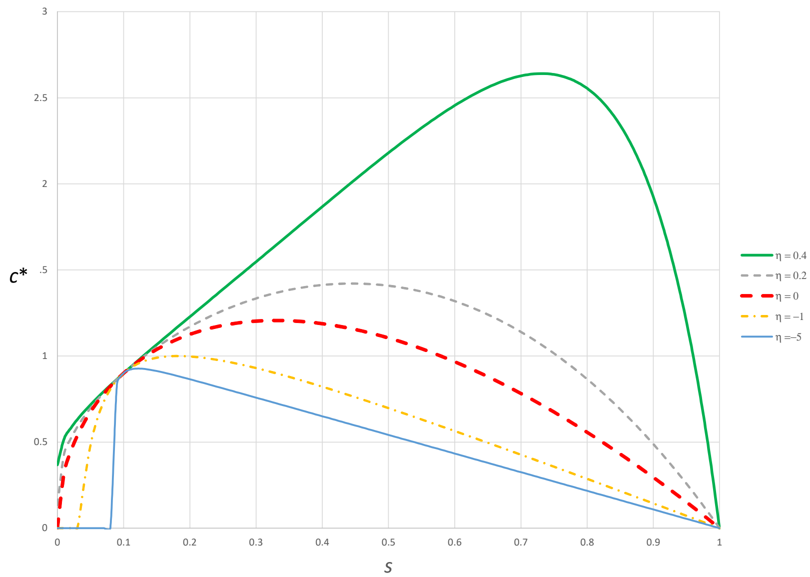

Figure 1: Graphs of steady state consumption as a function of the savings rate s for various choices of η and with $\mu = 0.33$ and $\xi = 0.1$.

We need the current capital-output ratio to pin down $k(0)$. There is less agreement about this because researchers often estimate capital-output ratios separately for state-owned enterprises and domestic private enterprises. Song et al. (2011) report ratios of 1.75 and 0.67 respectively for these two types of enterprises. Since the state-owned enterprises dominate the Chinese economy, a value of 1.5 seems reasonable as an average K/Y for the whole economy. Chen and Wen (2017), instead, report an average rate of profit of $r = 0.2$. Given a share of capital of 0.5 and a depreciation rate of 0.1, this implies $K/Y = 1.67$. To accommodate the imprecision of these estimates, we consider two values, 1.5 and 1.75, that roughly contain these various estimates of K/Y.

Since we do not have data with which to calibrate the elasticity of substitution between capital and labor, we proceed by considering different choices of η, which relates to this elasticity via (59). Using (58), we can rewrite (65) as

$$\alpha(k(0)) = \mu \left(\frac{K(0)}{Y(0)} \right)^{\eta} \tag{81}$$

since $\frac{K}{Y} = \frac{k}{f(k)}$. This allows us to calibrate μ. Then we compute s_{gr} via (67), which we plot as a function of η for both choices of K/Y in Fig. 2. We restrict attention to η such that $\mu, s_{gr} \in (0, 1)$. Since $\eta > -2$ is necessary to satisfy both of these conditions, we always have $\sigma > 0.2$, so $\widetilde{s} > s_{gr}$ iff s_{gr} satisfies (77).

The optimal ratio of the buffer between the target savings rate and the Golden rule savings rate to the noise variance is plotted for the two choices of K/Y in Fig. 3. This ratio is strictly decreasing in η and vanishes for $\eta = 0$ since, as we discussed in Section 4, steady-state consumption as a function of the savings rate is exactly symmetric around $s = 0.5$ if the production function is Cobb-Douglas and the share of capital is one half. $\eta = 0$ is also the dividing point between where (77) is and is not satisfied. So dynamic inefficiency will be optimal for $\eta < 0$. Thus the recent average savings rate of 43.3% could conceivably be optimal if η (or σ) are low enough so that $s_{gr} < 0.433$ or if $\eta > 0$. If $K/Y = 1.5$, we would need either $\sigma < 0.788$ or $\sigma > 1$. If $K/Y = 1.75$, we would need either $\sigma < 0.726$ or $\sigma > 1$.

As Fig. 3 shows, the optimal buffer to variance ratio is less than ten in magnitude for almost the entire feasible parameter space of η. If the standard deviation of ε is on the order of a few percentage points, the variance of ε will be on the order of 0.001, so the optimal buffer would typically be smaller than a percentage point.

We can discipline our calibration if we assume China is planning optimally and the variation in the observed savings rate in our sample is entirely due to variation in ε, in which case we obtain $\tau^2 = 0.000683$. Though the optimal buffer to variance ratio gets very large in magnitude as η approaches the upper limit where $s_{gr} \to 1$, it does not get large enough to have $|\Delta s| > 0.5$. Thus the observed average savings rate can only be optimal if capital and labor are complementary. If $K/Y = 1.5$, we need $\sigma = 0.788$. If $K/Y = 1.75$, we need $\sigma = 0.726$. In both of these cases, the Golden Rule steady state is 43.2%, so dynamic inefficiency is optimal although the degree of inefficiency is quite small.

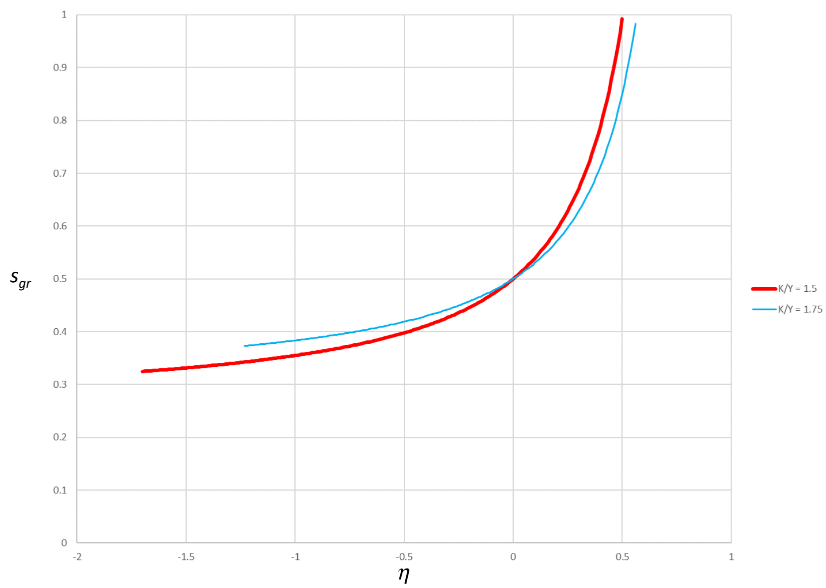

Figure 2: Golden Rule savings rate as a function of η for calibrations with K/Y equal to 1.5 or 1.75.

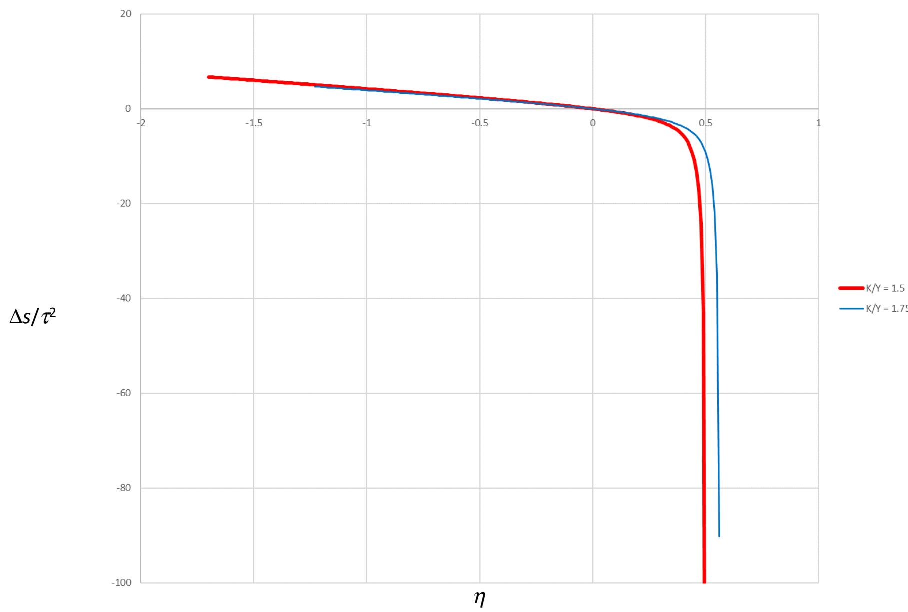

Figure 3: Ratio of optimal savings buffer to savings rate variance as a function of η for calibrations with K/Y equal to 1.5 or 1.75.

7 Concluding Remarks

We posit that it is an unnecessary stretch to model economies that are not associated with liberal democracies as standard neoclassical models featuring decentralized households that only coordinate their decisions via market-determined prices. Such economies can be described more parsimoniously with a Solow model that endogenizes the saving rate according to the principles of the governing regime. For the case of China, this approach can explain its high savings rate of 40% without any exotic assumptions about household preferences.

In a model with centralized decision making, some of the most hallowed results from decentralized models need not carry over if the social planner does not have perfect control over the economy. We find that a social planner who wants to maximize long-run aggregate consumption ought not to balk at violating the neoclassical rule that you should never save more than the income share of capital. The concept of dynamic inefficiency only makes sense for a hypothetical social planner that is omnipotent. Real-world social planners are not so powerful. Unlike individual households, social planners will be held responsible for economywide disasters, so they have to hedge against them. If consumption is an asymmetric function of the realized savings rate in the vicinity of the Golden Rule savings rate that maximizes consumption, the social planner will want to adjust the savings rate in the direction such that the falloff in consumption is less steep. We establish a necessary and sufficient condition on the production function under which it is better to oversave. As capital and labor become less substitutable, dynamic inefficiency becomes optimal for a larger subset of the parameter space since saving becomes more essential to continue production. Calibrations of China that match recent savings data satisfy the condition for dynamic inefficiency to be optimal.

In this initial treatment of the subject of precautionary social planning, we are effectively assuming the social planner decides on a thousand-year plan. We model the choice of a permanent savings rate based on its effects on consumption in the long run. In future work, we will be studying how the social planner ought to modify the savings rate target during the transition to the steady state. This will be analogous to setting a sequence of five-year plans, as China does in practice.

Of course, the primary criticism of social planners is not that they do not accumulate enough capital but that they do not accumulate enough of the right kinds of capital. If we generalize the present model to allow for multiple types of capital and/or multiple consumption goods, we could look separately at the effects of the underaccumulation of capital versus the misallocation of capital. We are not aware of any serious attempt to quantify which of these errors is more costly. However, we may soon get empirical data on the costs of these different errors from the natural experiment of how well China competes with the West through the 21st Century.

This paper may serve as a cautionary tale for Western countries that have previously prospered in an equilibrium where they all save at a relatively low rate. If they want to continue to compete with China in geopolitical terms, that strategy may not be tenable in the long run. The difficulty for Western countries is that, while the gains from increasing the saving rate in terms of higher GDP will be large, the gains in sustainable consumption will be quite modest. China has already made the sacrifice of raising its savings rate and now can slowly begin to enjoy the benefits of that sacrifice. Persuading households in Western countries to sacrifice 10-20% of their current consumption to put future generations on a higher growth path where they consume 1-2% more will be an extremely tough sell. Moreover, even if we do make this sacrifice, there will forever after be a huge temptation to squander it by consuming the surge in output and then resuming a path of low saving. A more feasible strategy for Western governments may be to ramp up public investment, for instance by passing some version of the Build Back Better package. This in turn might set an example for households to save more.

References

[1] Cass, D., (1965), "Optimal Growth in an Aggregative Model of Capital Accumulation," *Review of Economic Studies* 32: 233-240.

[2] Chen, Kaiji and Yi Wen, (2017), "The Great Housing Boom of China," *American Economic Journal: Macroeconomics* 9(2): 73-114.

[3] Cukierman, Alex, (2002), "Are Contemporary Central Banks Transparent About Economic Models and Objectives and What Difference Does It Make?" *Federal Reserve Bank of St. Louis Review* 84(4): 15-35.

[4] Diamond, Peter A., (1965), "National Debt in a Neoclassical Growth Model," *American Economic Review* 55: 1126-1150.

[5] Feigenbaum, James A., (2019), "A Nonparametric Formula Relating the Elasticity of a Factor Demand to the Elasticity of Substitution," *Theoretical Economics Letters* 9: 240-246.

[6] Feigenbaum, James A. and Frank N. Caliendo, (2010), "Optimal Irrational Behavior in Continuous Time," *Journal of Economic Dynamics and Control* 34: 1907-1922.

[7] Feigenbaum, James A., Frank N. Caliendo, and Emin Gahramanov, (2011), "Optimal Irrational Behavior," *Journal of Economic Behavior and Organization* 77: 286-304.

[8] Feigenbaum, James A., Emin Gahramanov, and Xueli Tang, (2013), "Is It Really Good to Annuitize?" *Journal of Economic Behavior and Organization* 93: 116-140.

[9] Friedman, Milton, (1953), *Essays in Positive Economics* (University of Chicago Press: Chicago).

[10] Garriga, Carlos, Aaron Hedlund, Yang Tang, and Ping Wang, (2021), "Rural-Urban Migration, Structural Transformation, and Housing Markets in China," Utah Macro Workshop Presentation.

[11] He, Hui, Lei Ning, and Dongming Zhu, (2019), "The Impact of Rapid Aging and Pension Reform on Savings and the Labor Supply: The Case of China," Working Paper.

[12] Koopmans, T., (1965) "On the Concept of Optimal Growth," in *The Econometric Approach to Development Planning* (Rand-McNally: Chicago).

[13] Leland, Hayne E., (1968), "Saving and Uncertainty: The Precautionary Demand for Saving," *Quarterly Journal of Economics* 82: 465-473.

[14] Marx, Karl, (1875), "Critique of the Gotha Program," https://www.marxists.org/archive/marx/works/download/Marx_Critque_of_the_Gotha_Programme.pdf.

[15] Ramsey, F. P., (1928), "A Mathematical Theory of Saving," *Economic Journal* 38: 543-559.

[16] Sandmo, A., (1970), "The Effect of Uncertainty on Saving Decisions," *Review of Economic Studies* 37: 353-360.

[17] Schot, Steven H., (1978), "Geometry of the Third Derivative," *Mathematics Magazine*, 51(5): 259-275.

[18] Solow, Robert M., (1956), "A Contribution to the Theory of Economic Growth," *Quarterly Journal of Economics* 70: 65-94.

[19] Song, Zheng, Kjetil Storesletten, and Fabrizio Zilibotti, (2011), "Growing Like China," *American Economic Review* 101: 196-233.