Perspectives on Present and Future Bias*

James Feigenbaum[†] Sepideh Raei[‡]

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Abstract

Present and future bias are a form of time-inconsistency in individuals' behavior toward trade-offs between consumption in the near future and far future. They are usually modeled in the literature with a relative discounting function, and the most common example in discrete time is a quasihyperbolic functional form. However, generalizing the concept of present and future bias beyond the quasihyperbolic case is not straightforward. In this paper, we propose a general representation of the discounting function in which the discount factor deviates from the exponential case through a perturbing parameter called the *future weighting factor*. The advantage of this representation is the simplicity of defining present bias in terms of all positive and strictly increasing future weighting factors whereas future bias occurs with all negative and strictly decreasing future weighting factors. We derive necessary and sufficient conditions on the future weighting factors to have a concave (convex) log consumption profile as well as necessary and sufficient conditions under which the consumption profile determined in the first period of life (almost) Pareto dominates the consumption profiles that are chosen in equilibrium. While a present (future) bias is a necessary condition for the former property, it is not a sufficient condition for either property.

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[†]Utah State University, John Huntsman School of Business, Utah, United States; james.feigenbaum@usu.edu.

[‡]Utah State University, John Huntsman School of Business, Utah, United States; sepideh.raei@usu.edu.

1 Introduction

Present bias is a form of time-inconsistency in which individuals are more impatient in trade-offs between the present and the immediate future as compared to trade-offs between equivalent intervals of time in the more distant future. An individual acting under this bias who might have been inclined to postpone a future payoff when the options of when to take it were all far in the future will become more inclined to take the payoff at the first opportunity when this opportunity gets closer to the present. Present bias, which is viewed as a form of misoptimization that accounts for a range of behavioral "mistakes," e.g. undersaving for retirement, has yielded a large literature that emphasizes the potential for policies like forced pensions or retirement saving subsidies to protect against or correct such mistakes.

Even though the idea of present bias is not new, it really took hold in economics following David Laibson's dissertation (Laibson (1994)), so the literature has blossomed in the past twenty-five years. Research has led to a much better theoretical understanding of present bias: when and how to apply it, and which assumptions are appropriate in different contexts (for a survey on present bias see O'Donoghue and Rabin (2015)). We can model present bias in terms of a discount function. However, unlike the discount function in a canonical model with exponential discounting, a discount function that exhibits present bias will be a function of the time to consumption from the present as opposed to the absolute time at which the consumption occurs.

In formal terms, suppose that intertemporal preferences from the perspective of period t can be represented by $U_t = \sum_{s=t}^{T} D_{s-t} u_s$, where u_s is instantaneous utility experienced in period s and D_x reflects the discounting associated with a delay of $x \in \{0, 1, 2, ...\}$. A common example in which the concept of present bias is readily discernible is the β-δ or "quasihyperbolic" functional form.

$$D_x = \begin{cases} 1 & \text{if } x = 0 \\ \beta\delta^x & \text{if } x > 0. \end{cases} \tag{1}$$

If $\beta = 1$, this reduces to an exponential discounting function, in which case the optimal plan at $t = 0$ will remain the optimal plan throughout the lifecycle. For $\beta \in (0, 1)$, the utility from consumption at all periods after the present are discounted by the factor β, and the difference $1 - \beta$ is a measure of present bias. The optimal plan at $t = 0$ will differ from the optimal plan later in life as the household will continually seek to advance consumption relative to what she originally planned. Conversely, if $\beta > 1$, the utility from consumption at all future periods would be magnified by the factor β, and $\beta - 1$ can be characterized as a measure of "future bias".

While the quasihyperbolic case only covers a measure-zero subset of the space of all possible discounting functions, because of their simplicity quasihyperbolic discount functions are often used as a proxy for other nonexponential discount functions. Indeed, the terminology of quasihyperbolic derives from this usage as an approximation to hyperbolic discount functions. If $\beta < 1$, the quasihyperbolic discount function will share with hyperbolic discount functions the property that the lifecycle profile of log consumption is concave. These discount functions also share another property to be further explained below: the household at most ages would prefer the consumption profile it would get if it could commit to its $t = 0$ plan to what it gets in equilibrium after accounting for its changing intertemporal preferences. On the other hand, a future-biased quasihyperbolic function with $\beta > 1$ will yield log consumption profiles that are convex, and the household would usually prefer the consumption profile it gets in equilibrium to what it would get if it could commit to its initial plan.

However, as we demonstrate in this paper, the language of "present" and "future" bias are not reliable predictors of these properties. A convex log consumption profile and almost Pareto dominance of the equilibrium allocations do not always arise in association with discount functions that one would naturally think of as future-biased. For example, a pure myopic discount function is a discount function that vanishes for delays beyond some horizon. Households with such a discount function do not care about consumption in the future beyond that horizon. Nevertheless, myopia yields properties consistent with a future-biased quasihyperbolic discount function rather than properties consistent with a present-biased quasihyperbolic discount function.[1]

The driving force behind the shape of the consumption profile and the preferences of the different selves can mainly be attributed to how our intertemporal valuations change in the far future rather than the immediate future. Our approach for showing this explicitly is by perturbing the discount function away from an exponential discount function, for which we know there is no time-inconsistency and the log consumption profile will be linear. More specifically, we focus on a general representation of a discount function $D_t = D_1^t(1 + \varepsilon_t)$ for $t > 1$, where ε_t is the extra weight we put on the discount factor t periods in the future.

[1] A myopic discount function actually exhibits both present and future bias, depending on the time horizon.

Thus we call ε_t the *future weighting factor*. In discrete time we can use linear approximations to simply interpret how the shape of the consumption profile and preferences between the equilibrium and commitment path are determined by the behavior of these future weighting factors.

This representation of the discount function also provides a straightforward way to understand the origin of a present bias, by having all ε_t be positive and strictly increasing for $t > 1$, or a future bias, by having all ε_t be negative and strictly decreasing for $t > 1$.[2] A present bias is a necessary, albeit not sufficient, condition to have a concave log consumption profile. Likewise, a future bias will be a necessary condition to have a convex log consumption profile.

Since positive future weights mean that the discount function will be higher than an exponential discount function as the delay time increases, we refer to this as "heavy future weighting", meaning the utility from future consumption will be weighted more heavily than for an exponential discount function. We refer to the case where the ε_t are negative as "light future weighting". In the case of a myopic discounting function the ε_t will all be -1 for sufficiently high t. The upshot is that a "future-biased" quasihyperbolic discount function has properties similar to a myopic discount function because both of these categories of discount functions put less weight on future consumption relative to an exponential discount function.

Another issue related to present and future bias that has been the focus of a relatively recent literature pertains to welfare analysis. Since an individual with time-inconsistent preferences, whether present- or future-biased, will choose a consumption profile that depends on the time of the choosing, it is not obvious which of these consumption profiles or the preferences at what period of life should be the reference point for welfare comparison. A common solution to this problem in the literature is to use the preferences of the initial self to evaluate welfare (see for example Laibson (1996), Laibson (1997), Laibson (1998), Laibson et al. (1998), O'Donoghue and Rabin (1999), O'Donoghue and Rabin, O'Donoghue and Rabin (2001) among many others). In fact, Caliendo and Findley (2019) show that commitment to the time-zero consumption plan can improve the objective function for all selves if the number of selves exceeds a certain threshold which turned out to be quite small in their setting.

Adding to this literature, the other contribution of this paper is to determine the conditions under which the commitment path will Pareto dominate the equilibrium path in

[2] For the case of a future bias we also need the additional requirement that $\varepsilon_t > -1$.

discrete time. We find that for the time-zero consumption profile to almost[3] Pareto dominate the consumption profiles that are chosen at each period of life we need the future weighting factor at the longest relevant delay, i.e. ε_T, to be sufficiently large. In other words, we find that if we put a heavy weight on the discount factor for the last period of life, as perceived during the first period of life, then at time zero the household will plan to consume a lot in the last period of life. To accomplish this the household will need to save for this terminal consumption at each successive period. However, this desire to consume so much at the end of life is fleeting and disappears for $t > 0$. Thus the household saves more over the course of the commitment path than it does on the equilibrium path, so the later selves would each (except possibly at $t = 1$) prefer the commitment path to the equilibrium path.

This condition is analogous to what we obtain in continuous time in Feigenbaum and Raei (2020). However, in discrete time it is possible to neatly express the condition for almost Pareto dominance in terms of a sequence of lower bounds on the terminal future weight ε_T that must all be satisfied. This emphasizes that what drives the welfare result is how the household values utility in the distant future rather than the immediate present. There is no simple interpretation of the analogous condition in continuous time because lifetime utilities are integrals rather than sums, so we can only isolate the effect of ε_T to establish when the terminal self would prefer the commitment path over the equilibrium path. On the other hand, in continuous time Feigenbaum and Raei (2020) are able to prove without approximations that the condition for the household to have a strictly concave log consumption profile is a sufficient condition for the terminal self to prefer the commitment path. Here in discrete time we are able to prove here to first order only for small life spans that a strictly concave log consumption profile implies almost Pareto dominance of the commitment path since the proof becomes more complicated with each additional period of life.

This paper is organized in the following way. Section 2 describes the model environment including the general format for the discount function. Section 3 develops the condition on the discount function for a concave or convex log consumption profile. Section 4 drives the condition on the discount function under which, commitment to the initial plan would almost Pareto dominate the equilibrium plan, and in section 5 we compare this Pareto dominance condition with the condition for concavity/convexity of the log consumption profile. Section

[3]In discrete time, the difference in welfare between the equilibrium and commitment paths at $t = 1$ is closely related to that difference at $t = 0$. Since the commitment path must always dominate the equilibrium path at $t = 0$, what happens at $t = 1$ can deviate from what happens over the rest of the life span.

6 concludes.

2 Model environment

We focus on a finite-horizon life-cycle model in which households live for $T+1$ periods. The household earns income $y_t \geq 0$ at age t for $t = 0, ..., T$, which can be consumed c_t or saved as k_{t+1} at a fixed gross interest rate $R \geq 0$.

2.1 Household optimization problem

At time t, a household with existing saving k_t maximizes

$$U_t = \sum_{s=t}^{T} D_{s-t} \ln c_{s|t} \tag{2}$$

subject to

$$c_{s|t} + k_{s+1|t} = y_s + R k_{s|t}, \quad s = t, \ldots, T, \tag{3}$$

where $D_t \geq 0$ is the discount function, and $c_{s|t}$ and $k_{s+1|t}$ are consumption and saving at period s as planned in period t[4]. Note that the household will solve this problem with $k_{t|t} = k_t$ and $k_{T+1|t} = 0$. To simplify notation, we will assume the household begins with $k_0 = 0$.[5]

Let us define

$$h_t = \sum_{s=t}^{T} \frac{y_s}{R^{s-t}}, \tag{4}$$

which represents the present value of the income stream from period t onward. Note that

$$h_t = y_t + \sum_{s=t+1}^{T} \frac{y_s}{R^{s-t}} = y_t + \frac{h_{t+1}}{R} \tag{5}$$

[4]The results are not qualitatively different for other CRRA utility functions, but they are more complicated so we only consider the logarithmic case. In solving the model we will proceed as though the household is naive about its time-inconsistency and does not know it will revise its plans as its preferences change. We could alternatively assume that the household is sophisticated about its time-inconsistency. However, with logarithmic period utility, the equilibrium path (and the commitment path in Section 4) will be the same under both assumptions, so there is no loss of generality. For more discussion see Marin-Solano and Navas (2009).

[5]Our results easily generalize if the household is endowed with savings or debt at birth.

for $t < T$. We can combine the period budget constraints from t to T into a lifetime budget constraint as of t:

$$\sum_{s=t}^{T} \frac{c_{s|t} + k_{s+1|t}}{R^{s-t}} = \sum_{s=t}^{T} \frac{y_s + Rk_{s|t}}{R^{s-t}}.$$

Using (4) and (5), this simplifies to

$$\sum_{s=t}^{T} \frac{c_{s|t}}{R^{s-t}} = h_t + Rk_t \tag{6}$$

The Lagrangian of the household problem at t can then be written as

$$L_t = \sum_{s=t}^{T} \left[D_{s-t} \ln c_{s|t} - \frac{\lambda_t c_{s|t}}{R^{s-t}} \right] + \lambda_t [h_t + Rk_t]. \tag{7}$$

Therefore, the first order condition (FOC) with respect to consumption will be

$$\frac{\partial L_t}{\partial c_{s|t}} = \frac{D_{s-t}}{c_{s|t}} - \frac{\lambda_t}{R^{s-t}} = 0 \tag{8}$$

The initial consumption plan $c_{s|0}$ that is determined at $t = 0$, the first period of life, will be referred to hereafter as the **commitment path**. Note, however, that unless the discount function is exponential the household will only follow the initial plan at $t = 0$. Indeed, at each period t of life, the household will choose a new plan $c_{s|t}$, but only the choice of consumption at t, $c_t = c_{t|t}$, will adhere to this plan. As the household progresses from period to period, its preferences will unexpectedly change since we are assuming that the household is naive about the change in its future preferences. When it gets to $t + 1$, it will then have saving $k_{t+1} = k_{t+1|t}$, but it will solve (7) anew, updated to $t + 1$. The resulting consumption profile c_t, determined at each period t, will be referred to as the **equilibrium path**.

While the FOC (8) for $t = 0$ governs the whole commitment path for consumption $c_{s|0}$ from $s = 0, \ldots, T$, along the equilibrium path only the FOC with $s = t$ will actually matter. This simplifies to

$$\frac{D_{t-t}}{c_{t|t}} - \frac{\lambda_t}{R^{t-t}} = 0,$$

so we have

$$\lambda_t = \frac{1}{c_t}$$

since $c_t = c_{t|t}$ and $D_0 = 1$. The future plan $c_{s|t}$ at t is only relevant to the extent that it determines the Lagrange multiplier λ_t. Generalizing (8), we obtain

$$c_{s|t} = \frac{D_{s-t}R^{s-t}}{\lambda_t} = D_{s-t}R^{s-t}c_t. \tag{9}$$

Inserting these into the lifetime budget constraint (6), we get

$$\sum_{s=t}^{T} \frac{D_{s-t}R^{s-t}c_t}{R^{s-t}} = h_t + Rk_t,$$

which reduces to

$$c_t = \frac{h_t + Rk_t}{\sum_{s=t}^{T} D_{s-t}}. \tag{10}$$

Hence, on the equilibrium path, the budget constraint on period t can be written as

$$k_{t+1} = k_{t+1|t} = y_t + Rk_t - c_t = y_t + Rk_t - \frac{h_t + Rk_t}{\sum_{s=t}^{T} D_{s-t}}. \tag{11}$$

We can use this to calculate an effective Euler equation along the equilibrium path. Combining (5) and (11), we get,

$$\begin{aligned} h_{t+1} + Rk_{t+1} &= R\left(\frac{h_{t+1}}{R} + y_t + Rk_t - \frac{h_t + Rk_t}{\sum_{s=t}^{T} D_{s-t}}\right) \\ &= R\left(h_t + Rk_t - \frac{h_t + Rk_t}{\sum_{s=t}^{T} D_{s-t}}\right) \\ &= R\left(\frac{\sum_{s'=t+1}^{T} D_{s'-t}}{\sum_{s=t}^{T} D_{s-t}}\right)(h_t + Rk_t). \end{aligned}$$

Updating (10) to $t+1$, consumption at $t+1$ is

$$c_{t+1} = R\left(\frac{\sum_{s'=t+1}^{T} D_{s'-t}}{\sum_{s=t}^{T} D_{s-t}}\right)\frac{h_t + Rk_t}{\sum_{s=t+1}^{T} D_{s-t-1}}$$

Applying (10) again in its original form, the Euler equation in equilibrium for a general discounting function D_t with log utility is

$$c_{t+1} = R\frac{\sum_{s'=t+1}^{T} D_{s'-t}}{\sum_{s=t+1}^{T} D_{s-t-1}}c_t. \tag{12}$$

In the special case of an exponential discount function $D_t = \delta^t$, the ratio

$$\mathcal{D}_t = \frac{\sum_{s'=t+1}^{T} D_{s'-t}}{\sum_{s=t+1}^{T} D_{s-t-1}} \tag{13}$$

simplifies to the constant δ, and we get back the familiar Euler equation $c_{t+1} = \delta R c_t$. More generally, though, for a nonexponential discount function, the inverse ratio $\mathcal{D}_t^{-1}$ measures the gross rate of change in the sum of the discount functions relevant at periods $t+1$ to T as the household moves from t to $t+1$. That is to say the change from the sum $D_1 + \cdots + D_{T-t}$ applicable at t to the sum $1 + \cdots + D_{T-t-1}$ applicable at $t+1$. The richer consumption dynamics that can be obtained in equilibrium with nonexponential discounting functions stems entirely from the deviation of the $\mathcal{D}_t$ from a constant, which will depend on how the discount function D_t deviates from an exponential function.

2.2 Future Weighting Discount Function

As mentioned before, given a discount function $D_t \geq 0$ for $t = 0, ..., T$, we can define the "future weighting factor" ε_t via

$$D_t = D_1^t(1 + \varepsilon_t). \tag{14}$$

where D_1 is the discount factor for one period ahead. This future weighting factor basically captures the extra (or diminished, if negative) weight that we put on the discounting t periods in the future. Since we normalize $D_0 = 1$, by definition we will have $\varepsilon_0 = \varepsilon_1 = 0$.

For example, we can represent a quasihyperbolic discount function by setting $\varepsilon_t = \beta^{1-t} - 1$;

$$D_t = \beta\delta^t$$

for $t > 0$ with $D_0 = 1$. Since

$$D_1 = \beta\delta,$$

ε_t can be calculated as

$$\frac{D_t}{D_1^t} = \frac{\beta\delta^t}{\beta^t\delta^t} = \beta^{1-t} = 1 + \varepsilon_t.$$

Hence

$$\varepsilon_t = \beta^{1-t} - 1. \tag{15}$$

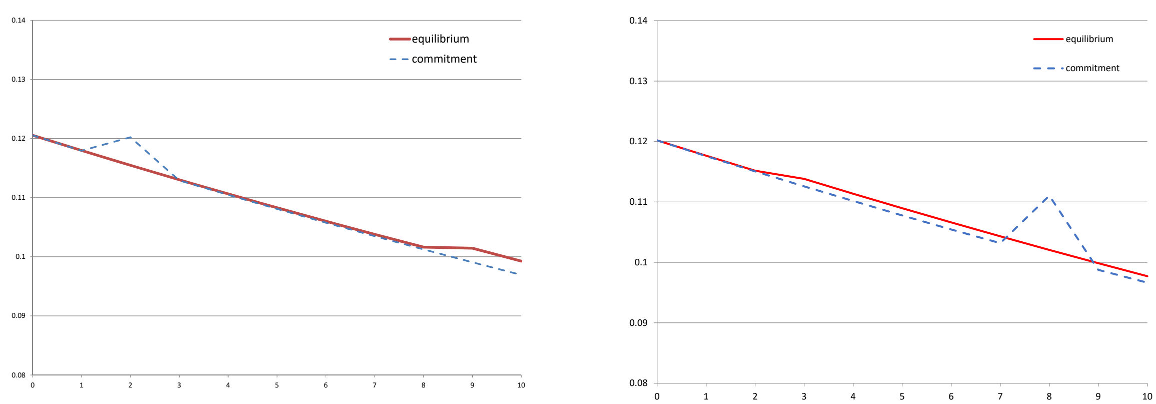
Likewise, for a myopic discounting function that vanishes for $t \geq t^*$, we have $\varepsilon_t = -1$ for

$t \geq t^*$.

Note that if $\varepsilon_t = 0$ for all t, the discount function will be exponential. Thus we can think of the future weighting factor, ε_t, as the parameter that measures the discount function's deviation from an exponential at the delay t. If $\varepsilon_t > 0$, the discount factor of t periods in the future will be higher than an exponential discount factor. We call this "heavy future weighting" since utility from consumption t periods in the future will be weighted more heavily than it would be under an exponential discount function. Conversely, with $\varepsilon_t < 0$ the discount function exhibits "light future weighting" relative to an exponential discount function.

To have a better understanding of the role of ε_t in determining consumption behavior, figure 1 compares the consumption profile under the commitment path and the equilibrium path for a ten period model, $T = 10$. We consider two cases to demonstrate the role of an individual ε_t. First, we have a discount function for which ε_t is zero for all t except $t = 2$. Second, we have a discount function for which ε_t is zero for all t except $t = 8$.

Figure 1: consumption profile, commitment path and equilibrium path



a . only $\varepsilon_2 > 0$

b . only $\varepsilon_8 > 0$

Note: on both graphs the horizontal axis is time and vertical access is the consumption level at each period.

In both plots, the blue dashed line shows the commitment path and the red solid line shows the equilibrium path. In figure 1a, we see a spike in period two along the commitment path simply because $\varepsilon_2 > 0$ means that the household initially puts a higher weight on the utility from consuming two periods ahead compared to all other future periods. Hence, the spike at $t = 2$. Likewise, looking at figure 1b in which $\varepsilon_8 > 0$, the spike in the commitment path is at $t = 8$.

The effect of ε_t on the equilibrium path is much more subtle than for the commitment path. With $\varepsilon_2 > 0$, shown in figure 1a, the household continually plans to have high consumption two periods ahead, as happens at $t = 2$ on the commitment path. However, with each new period, she reoptimizes and pushes forward when she intends to have high consumption. This trend continues until the household arrives at period nine of her lifetime, at which point there no longer is a period two periods ahead. Consequently, the equilibrium consumption path is quite smooth, as it would be with exponential discounting, for $t < 9$. She does not realize this intended high consumption two periods ahead until she can no longer defer this consumption. From this point, all future periods are discounted with the same rate. Consumption jumps up in these last two periods as she finally consumes the saving she accumulated to finance the planned extra consumption two periods ahead.

The same intuition applies to figure 1b in which $\varepsilon_8 > 0$. There, the future period with a higher discounting factor disappears after the second period. That is the reason why the equilibrium consumption plan for $t \geq 3$ shifts upward. The high ε_8 disappears from her calculus once there no longer is a period eight periods ahead within her remaining time horizon. Consequently, she behaves like an exponential discounter thereafter, smoothing out over all the periods with $t \geq 3$ the extra consumption that she had previously intended, at $t = 2$, to save entirely for the last period.

A discount function exhibits present bias at $t > 0$ if it gives rise to the following type of preference reversal. Suppose for some allocation $\{c_t\}_{t=0}^{T}$, there exists $\Delta c_t > 0$ and $\Delta c_{t+1} \in (0, c_{t+1})$ such that the household would prefer at time 0 the original allocation over a forward-shifted allocation with c_t increased by Δc_t and c_{t+1} decreased by Δc_{t+1}. However, when the household gets to time t, it instead prefers the forward-shifted allocation over the original allocation. Thus the household would prefer not to shift consumption forward when the possibility of doing so is in the future, but it would opt to make that shift in the present. This is usually interpreted as the household putting an extra preference on consumption in the immediate present. Future bias at $t > 0$ is defined similarly except the preference reversal goes the other way. The household would prefer the forward-shifted allocation over the original allocation when t is in the future, and prefers the original allocation when it reaches time t. We say a discount function is present-biased (future-biased) if it exhibits present (future) bias at all $t > 0$.

Assuming $D_s > 0$ for all s, we can express the condition for preference reversals in terms of the perceived marginal rate of substitution between consumption at t and consumption at $t+1$ as of time $s \le t$:

$$m_s(t) = \frac{D_{t+1-s}u'(c_{t+1})}{D_{t-s}u'(c_t)}. \tag{16}$$

The household will prefer the forward-shifted allocation at time 0 and the original allocation at t if

$$D_t u'(c_t)\Delta c_t - D_{t+1}u'(c_{t+1})\Delta c_{t+1} < 0 < u'(c_t)\Delta c_t - D_1 u'(c_{t+1})\Delta c_{t+1},$$

which we can rearrange as

$$\frac{D_1 u'(c_{t+1})}{u'(c_t)} < \frac{\Delta c_t}{\Delta c_{t+1}} < \frac{D_{t+1}u'(c_{t+1})}{D_t u'(c_t)}.$$

Thus the household will have a present bias at t if $m_0(t) < m_t(t)$ since we can then find Δc_t and Δc_{t+1} such that $\frac{\Delta c_t}{\Delta c_{t+1}} \in (m_0(t), m_t(t))$. Since $\varepsilon_0 = \varepsilon_1 = 0$ by definition, we can express this condition in terms of future weighting factors as $\frac{1+\varepsilon_{t+1}}{1+\varepsilon_t} > 1$ or more simply as $\varepsilon_t < \varepsilon_{t+1}$.

A present-biased discount function will have strictly increasing and positive (for $t > 2$) future weighting factors. Conversely, a strictly positive and future-biased discount function will have strictly decreasing and negative (for $t > 2$) future weighting factors.

Note that a myopic discount function that is zero for t greater than equal to some $t^* > 1$ does not fit nicely into the categories of a present- or future-biased discount function because it does not satisfy the caveat that the D_t are all positive. There will be a future bias at $t^* - 1$ since at time zero the household would prefer not to consume anything at t^*, but its $(t^* - 1)$-utility is only defined if $c_{t^*} > 0$. On the other hand, there will be a weak present bias at $t \ge t^*$ since at time zero the household will be indifferent between how it allocates consumption between t and $t+1$. However, at time t the household will prefer to have more consumption at t if $D_1 > 0$.

3 Curvature of the optimal consumption profile

As we discussed in the previous section, ε_t is the parameter that controls the discounting weight of future periods, with a positive ε_t interpreted as a "heavy future weighting" and a negative ε_t as a "light future weighting". In this section, we explore how the value of the future weighting factor, ε_t, determines the curvature of the optimal consumption profile of the household. More precisely, we establish a sufficient condition on ε_t under which the consumption profile would be concave (convex).

As the first step, we will rewrite the Euler equation in terms of the future weighting discount function. Replacing the general form of discounting function D_t in the household's Euler equation (12) with the form involving the future weighting discounting function (14) gives us

$$c_{t+1} = D_1 R \frac{\sum_{s'=t+1}^{T} D_1^{s'}(1+\varepsilon_{s'-t})}{\sum_{s=t+1}^{T} D_1^{s}(1+\varepsilon_{s-t-1})} c_t. \tag{17}$$

In this still exact form, it is more apparent that the Euler equation reduces to the usual $c_{t+1} = D_1 R c_t$ when we have an exponential discounting function and $\varepsilon_2 = \varepsilon_3 = \cdots = \varepsilon_T = 0$.

Next let us focus on the first-order approximation to (17), disregarding all terms that are second or higher order in the ε_t. If we factor out the summation $\sum_{s=t+1}^{T} D_1^s$ from both the numerator and the denominator, the Euler equation becomes

$$c_{t+1} = D_1 R \frac{1 + \frac{\sum_{s'=t+1}^{T} D_1^{s'} \varepsilon_{s'-t}}{\sum_{j=t+1}^{T} D_1^{j}}}{1 + \frac{\sum_{s=t+1}^{T} D_1^{s} \varepsilon_{s-t-1}}{\sum_{l=t+1}^{T} D_1^{l}}} c_t. \tag{18}$$

The first-order Taylor expansion of $f(x) = \frac{1}{1+x}$ is

$$\begin{aligned} f(x) &= f(0) + f'(0)x + O(x^2) \\ &= 1 - x + O(x^2), \end{aligned}$$

where $O(g(x))$ represents an unspecified function smaller than $Mg(x)$ for some $M > 0$ in the limit as $x \to 0$. Therefore, we can rewrite (18) as

$$c_{t+1} = D_1 R \left[1 + \frac{\sum_{s'=t+1}^{T} D_1^{s'} \varepsilon_{s'-t}}{\sum_{s=t+1}^{T} D_1^{s}} - \frac{\sum_{s'=t+1}^{T} D_1^{s'} \varepsilon_{s'-t-1}}{\sum_{s=t+1}^{T} D_1^{s}} \right] c_t + O(\varepsilon^2),$$

which reduces to

$$c_{t+1} = D_1 R \left[1 + \frac{\sum_{s=t+1}^{T} D_1^{s} (\varepsilon_{s-t} - \varepsilon_{s-t-1})}{\sum_{s'=t+1}^{T} D_1^{s'}} \right] c_t + O(\varepsilon^2). \tag{19}$$

This represents the Euler equation to first order in the future weighting factor ε_t. Thus deviations of the Euler equation from the canonical Euler equation $c_{t+1} = D_1 R c_t$ for an exponential discounting function arise because of changes in the future weighting as the delay changes by one period. The effect of a change in future weighting s periods in the future is discounted, to first order, by D_1^s, so a change in the future weighting at short delays will have a bigger effect than a change at long delays.

Let us define

$$\Delta\varepsilon_t = \varepsilon_{t+1} - \varepsilon_t. \tag{20}$$

Then the Euler equation with this Δ notation is

$$c_{t+1} = D_1 R \left[1 + \frac{\sum_{s=t+1}^{T} D_1^s \Delta\varepsilon_{s-t-1}}{\sum_{s'=t+1}^{T} D_1^{s'}}\right] c_t + O(\varepsilon^2). \tag{21}$$

We will focus on the log consumption profile, which will be concave if $\log(\frac{c_{t+1}}{c_t})$ decreases with t. Taking the log of the Euler equation, for $t = 0, \ldots, T-1$,

$$\Delta \ln c_t = \ln c_{t+1} - \ln c_t = \ln D_1 R + \frac{\sum_{s=t+1}^{T} D_1^s \Delta\varepsilon_{s-t-1}}{\sum_{s'=t+1}^{T} D_1^{s'}} + O(\varepsilon^2). \tag{22}$$

Likewise,

$$\Delta \ln c_{t+1} = \ln c_{t+2} - \ln c_{t+1} = \ln D_1 R + \frac{\sum_{s=t+2}^{T} D_1^s \Delta\varepsilon_{s-t-2}}{\sum_{s'=t+2}^{T} D_1^{s'}} + O(\varepsilon^2). \tag{23}$$

Similarly, for $t = 0, ..., T-2$, we can also define the second-order difference

$$\Delta^2 \ln c_t = \frac{\sum_{s=t+2}^{T} D_1^s \Delta\varepsilon_{s-t-2}}{\sum_{s'=t+2}^{T} D_1^{s'}} - \frac{\sum_{s=t+1}^{T} D_1^s \Delta\varepsilon_{s-t-1}}{\sum_{s'=t+1}^{T} D_1^{s'}} + O(\varepsilon^2). \tag{24}$$

To have a concave log consumption profile we need $\Delta^2 \ln c_t < 0$ for $t = 0, ..., T-2$. Notice that the $\ln D_1 R$ in (22) and (23) vanishes from (24). All of the remaining terms in (24) are of first or higher order in the ε_t. Thus the log consumption profile with an exponential discounting function is exactly linear. Any deviation from linearity is driven by the future weighting factors.

We can simplify (24) by coalescing the two summations into one summation of the $\Delta\varepsilon_i$. We do this by letting $i = s - t - 2$, so $s = i + t + 2$, and $j = s - t - 1$, so $s = j + t + 1$. Then

$$\Delta^2 \ln c_t = \frac{\sum_{i=0}^{T-t-2} D_1^i \Delta\varepsilon_i}{\sum_{i'=0}^{T-t-2} D_1^{i'}} - \frac{\sum_{j=0}^{T-t-1} D_1^j \Delta\varepsilon_j}{\sum_{j'=0}^{T-t-1} D_1^{j'}} + O(\varepsilon^2),$$

which can be rearranged as

$$\Delta^2 \ln c_t = \frac{D_1^{T-t-1}}{(\sum_{i'=0}^{T-t-2} D_1^{i'})(\sum_{j'=0}^{T-t-1} D_1^{j'})} \sum_{i=0}^{T-t-2} D_1^i \Delta\varepsilon_i - \frac{D_1^{T-t-1} \Delta\varepsilon_{T-t-1}}{\sum_{j'=0}^{T-t-1} D_1^{j'}}.$$

This simplifies to

$$\Delta^2 \ln c_t = \frac{1 - D_1}{1 - D_1^{T-t}} D_1^{T-t-1} \left[\frac{\sum_{i=0}^{T-t-2} D_1^i \Delta\varepsilon_i}{\sum_{i'=0}^{T-t-2} D_1^{i'}} - \Delta\varepsilon_{T-t-1} \right] + O(\varepsilon^2).$$

As we mentioned above, for $\ln c_t$ to be strictly concave we need $\Delta^2 \ln c_t < 0$. This implies that to first order in ε, strict concavity of $\ln c_t$ requires

$$\Delta\varepsilon_{T-t-1} > \frac{\sum_{i=0}^{T-t-2} D_1^i \Delta\varepsilon_i}{\sum_{i'=0}^{T-t-2} D_1^{i'}}.$$

If this is true for $t = 0, \ldots, T-2$, we must have

$$\Delta\varepsilon_s > \frac{\sum_{i=0}^{s-1} D_1^i \Delta\varepsilon_i}{\sum_{i'=0}^{s-1} D_1^{i'}} \tag{25}$$

for $s = 1, \ldots, T-1$.

Since $\Delta\varepsilon_0 = \varepsilon_1 - \varepsilon_0 = 0$, by definition the condition for $s = 1$ implies $\varepsilon_2 = \Delta\varepsilon_1 > 0$. Note also that the condition (25) for $s = 1$ corresponds to the first-order condition for $\Delta^2 \ln c_{T-2} < 0$, i.e. for the log consumption profile to be strictly concave between periods $T-2$ and T. A positive future weighting two periods ahead means consumption growth between $T-1$ and T will be lower than between $T-2$ and $T-1$, producing a concavity in the log consumption profile at the end of life. Conversely, a negative future weighting will result in a convex log consumption profile between $T-2$ and T.

Proposition 1. *For the entire log consumption profile to be strictly concave (convex), the $\Delta\varepsilon_s$ from $s = 1, \ldots, T-1$ must be positive (negative). Consequently, the ε_s from $s = 2, \ldots, T$ must all be strictly increasing (decreasing) and therefore also positive (negative). In other words, a necessary condition for the log consumption profile to be strictly concave is that the discount function is present-biased. And, assuming $\varepsilon_t > -1$ for all t, a necessary condition for the log consumption profile to be strictly convex is that the discount function is future-biased.*

This proposition can be proved by induction. Suppose the $\Delta\varepsilon_i$ are positive for $s =$

$1, \ldots, s-1$. Then (25) implies $\Delta\varepsilon_s$ is also positive, and $\varepsilon_{s+1} = \varepsilon_s + \Delta\varepsilon_s > \varepsilon_s > 0$. Note also that each successive iteration of (25) is the necessary condition for the log consumption profile to be concave one period earlier. Thus the condition that

$$\varepsilon_T > \varepsilon_{T-1} + \frac{\sum_{i=0}^{T-2} D_1^i \Delta\varepsilon_i}{\sum_{i'=0}^{T-2} D_1^{i'}}$$

is the first-order condition that the log consumption profile is strictly concave between $t = 0$ and $t = 2$. Iterating backwards, each log consumption growth ratio will depend on one more difference $\Delta\varepsilon_s$ than the ensuing log consumption growth ratio, so $\Delta\varepsilon_s > 0$, or equivalently $\varepsilon_{s+1} > \varepsilon_s$ will be necessary to have the log consumption growth ratio decrease with time. One way to think about this result is that what often gets referred to as present bias is really a case of the future gradually mattering less as the future gets closer to the present. Therefore, the ε_t must grow with t because that implies the extra weight associated with a specific age gets smaller as we approach that age and the delay time gets shorter.

How fast must the future discounting weights grow to get a strictly concave log consumption profile? For a given ε_2, let us define a lower bound, $\Delta\underline{\varepsilon}_s$, on $\Delta\varepsilon_s$ such that the corresponding (25) must hold. For the case of $s = 1$, (26) evaluates straightforwardly to

$$\Delta\underline{\varepsilon}_2 = \frac{D_1}{1 + D_1}\varepsilon_2.$$

A necessary and sufficient condition for strict concavity from $t = T - 3$ to $t = T$ is that $\varepsilon_2 = \Delta\varepsilon_1 > 0 = \Delta\underline{\varepsilon}_1$ and $\Delta\varepsilon_2 > \Delta\underline{\varepsilon}_2$. We can thus iteratively define

$$\Delta\underline{\varepsilon}_{s+1} = \frac{\sum_{i=0}^{s} D_1^i \Delta\underline{\varepsilon}_i}{\sum_{i'=0}^{s} D_1^{i'}} = \frac{\sum_{i=1}^{s} D_1^i \Delta\underline{\varepsilon}_i}{\sum_{i'=0}^{s} D_1^{i'}}, \tag{26}$$

where $\Delta\underline{\varepsilon}_1 = \varepsilon_2$ is given. If $\Delta\varepsilon_i > \Delta\underline{\varepsilon}_i$ for $i = 2, \ldots, s$ are all necessary conditions for strict concavity, then $\Delta\varepsilon_{s+1} > \Delta\underline{\varepsilon}_{s+1}$ will also be a necessary condition for strict concavity. To put it another way, if the $\Delta\varepsilon_s = \Delta\underline{\varepsilon}_s$ for $s = 2, \ldots, T-1$ and $\varepsilon_2 > 0$, the log consumption profile will be linear from $t = 0$ to $t = T - 1$ and strictly concave between $t = T - 2$ and $t = T$.

Proposition 2. *For $t \geq 2$,*

$$\Delta\underline{\varepsilon}_t = \frac{D_1}{1 + D_1}\varepsilon_2.$$

The proof follows by induction. If it is true for $2, \ldots, s$,

$$\begin{aligned}
\Delta\underline{\varepsilon}_{s+1} &= \frac{\sum_{i=1}^{s} D_1^i \Delta\underline{\varepsilon}_i}{\sum_{i'=0}^{s} D_1^{i'}} = \frac{D_1\varepsilon_2 + \sum_{i=2}^{s} D_1^i \Delta\underline{\varepsilon}_2}{\sum_{i'=0}^{s} D_1^{i'}} \\
&= \frac{D_1 + \frac{D_1}{1+D_1}\sum_{i=2}^{s} D_1^i}{\sum_{i'=0}^{s} D_1^{i'}}\varepsilon_2 \\
&= \frac{D_1}{1+D_1}\frac{\sum_{i=0}^{s} D_1^i}{\sum_{i'=0}^{s} D_1^{i'}}\varepsilon_2 = \frac{D_1}{1+D_1}\varepsilon_2.
\end{aligned}$$

Therefore, a necessary condition for the log consumption profile to be concave is to have $\varepsilon_2 > 0$ and

$$\Delta\varepsilon_s \geq \Delta\underline{\varepsilon}_s = \frac{D_1}{1+D_1}\varepsilon_2 \tag{27}$$

for $s = 2, \ldots, T-1$. This implies that a necessary condition for weak concavity of log consumption profile is that ε_2 must be positive and the ε_t must grow, but they need only grow linearly with a slope greater than $\frac{D_1}{1+D_1}\varepsilon_2$. It is only a necessary condition because as the $\Delta\varepsilon_i > \Delta\underline{\varepsilon}_i$ for $i = 2, \ldots, s$, $\Delta\varepsilon_{s+1}$ must likewise be greater than $\Delta\underline{\varepsilon}_{s+1}$ to satisfy its concavity bound. If the future discounting weights grow faster than linearly at short delays, they must continue to grow faster than linearly at longer delays to maintain strict concavity over the whole lifespan.

We can derive concavity conditions for two popular forms of discount function, the quasi-hyperbolic and hyperbolic discount functions, and show that they are satisfied to first order.

Example: quasihyperbolic discount function

First, let us consider the quasihyperbolic discounting function

$$D_s = \beta\delta^s$$

in which

$$\beta = 1 - \eta \tag{28}$$

for small $\eta > 0$. Then (15) gives

$$\varepsilon_s = (1-\eta)^{1-s} - 1 = (s-1)\eta + O(\eta^2) \tag{29}$$

for $s \geq 1$. Therefore,

$$\Delta\varepsilon_s = \eta + O(\eta^2) \tag{30}$$

for $s \geq 1$. From (25), the concavity bound on $\Delta\varepsilon_s$ is

$$\frac{\sum_{i=0}^{s-1} D_1^i \Delta\varepsilon_i}{\sum_{i'=0}^{s-1} D_1^{i'}} = \eta(1 - \frac{1}{\sum_{i=0}^{i=s-1} D_1^i}) + O(\eta^2) < \eta + O(\eta^2).$$

Thus the concavity bound will clearly be satisfied by a quasihyperbolic discounting function to first order in η. Note that in this case

$$\Delta\underline{\varepsilon}_s = \frac{D_1}{1 + D_1}\varepsilon_2 = \frac{\beta\delta}{1 + \beta\delta}\eta + O(\eta^2),$$

which is less than $\Delta\varepsilon_s$ to first order in η. Thus with quasihyperbolic discounting, the future discounting weights grow approximately linearly and faster than is required to satisfy the necessary condition for concavity.

Example: hyperbolic discount function

Second, we consider a hyperbolic discounting function,

$$D_s = \frac{1}{1 + \eta s}$$

for small $\eta > 0$ in which

$$D_1 = \frac{1}{1 + \eta}$$

and

$$\varepsilon_s = \frac{(1 + \eta)^s}{1 + \eta s} - 1 = \eta^2 \frac{s(s-1)}{2} + O(\eta^3) \tag{31}$$

for $s \geq 0$.

Then

$$\Delta\varepsilon_s = \varepsilon_{s+1} - \varepsilon_s = \frac{s(s+1)}{2}\eta^2 - \frac{s(s-1)}{2}\eta^2 + O(\eta^3) = s\eta^2 + O(\eta^3). \tag{32}$$

From (25), the concavity bound on $\Delta\varepsilon_s$ is

$$\frac{\sum_{i=0}^{s-1} D_1^i \Delta\varepsilon_i}{\sum_{i'=0}^{s-1} D_1^{i'}} = \eta^2 \frac{\sum_{i=1}^{s-1} s}{s} + O(\eta^3) = \eta^2 \frac{s-1}{2} + O(\eta^3).$$

Comparing this to (32), we see that a hyperbolic discounting function also satisfies the concavity bounds to first order. With hyperbolic discounting, the future discounting weights grow quadratically, so the necessary condition for concavity is more than satisfied.

4 Pareto dominance of the commitment path

In previous sections, we described the household problem with a discounting function that depends on the time to consumption from the present rather than the absolute time when the consumption occurs. Such a household has time-inconsistent preferences and therefore, as Strotz (1955) noted, the marginal rate of substitution between consumption at different times depends on when the household is evaluating the utility from these consumptions. Consequently, the household at different ages will value consumption plans differently. This multiplicity of selves can substantially complicate welfare analysis.

A common solution to tackle this complication in the literature is to use the preferences of the initial self to evaluate welfare. See, for example, (Laibson, 1997, 1996), Laibson et al. (1998), and (O'Donoghue and Rabin, 1999, 2001). This approach does have its criticisms however. Dewatripont et al. (2004) states that there is "no normative foundation" for equating welfare with time-zero preferences.

A more recent literature explores conditions under which committing to the initial plan of the time-zero self improves the welfare of all selves over the life cycle as compared to what they would obtain in equilibrium, providing a justification for singling out the preferences of the time-zero self. Caliendo and Findley (2019) show that with quasihyperbolic discounting commitment to the time-zero consumption plan can improve the objective function for all selves if the number of selves exceeds a certain threshold which turned out to be quite small in their setting. Expanding upon this result, Feigenbaum and Raei (2020) show in a continuous-time setup conditions under which commitment to the initial plan will be Pareto improving for all the different selves. However, because almost all of these conditions involve integrals of future weighting factors instead of sums, they are difficult to interpret. In this section we obtain a similar but simpler results in discrete time, formulating a condition on the future weighting discount function under which committing to the initial plan will almost Pareto dominate the equilibrium plan.

The utility realized in equilibrium as of time t is simply the realized value of the household's objective function at time t:

$$U_t^* = \sum_{s=t}^{T} D_{s-t} \ln(c_s) \tag{33}$$

In contrast, the commitment utility at time t is

$$U_t^c = \sum_{s=t}^{T} D_{s-t} \ln(c_{s|0}), \tag{34}$$

which is what you obtain if you insert the original $t = 0$ consumption path into the objective function at time t. What concerns us most is the difference ΔU_t in realized utility between the equilibrium plan and the original plan at time t:

$$\Delta U_t = U_t^* - U_t^c = \sum_{s=t}^{T} D_{s-t} \ln\left(\frac{c_s}{c_{s|0}}\right). \tag{35}$$

Note that if $\Delta U_t > 0$, then following the equilibrium consumption plan provides the household with a higher utility compared to the initial plan. Conversely, if $\Delta U_t < 0$, then committing to the initial plan is optimal. This is a general form and D_{s-t} can be replaced with any discounting function. For example with $D_t = \beta^t$ both the original plan and equilibrium plan coincide and $\Delta U_t = 0$, meaning that the household will be indifferent between the two. By definition, the commitment path must maximize lifetime utility at $t = 0$, so we must have $\Delta U_0 \leq 0$. In what follows, we will see that $\Delta U_1 = 0$ must always hold to first order in the future weighting factors. We will then investigate conditions on the future weighting factors under which the initial path would almost Pareto dominate the equilibrium path for the household to first order, i.e. $\Delta U_t < O(\varepsilon^2)$ for $t > 1$.

For the original plan $c_{t|0}$, the consumption at period t, as determined at period 0, can be written

$$c_{t|0} = D_t R^t c_0 = D_1^t (1 + \varepsilon_t) R^t c_0. \tag{36}$$

Let us define

$$c_t^0 = D_1^t R^t c_0. \tag{37}$$

Therefore

$$\ln c_{t|0} = \ln c_t^0 + \varepsilon_t + O(\varepsilon^2), \tag{38}$$

and we can simplify the utility obtained at time t from committing to the initial consumption plan as

$$U_t^c = \sum_{s=t}^{T} \left[D_{s-t} \ln(c_s^0) + D_1^{s-t} \varepsilon_s \right] + O(\varepsilon^2). \tag{39}$$

To compute the equilibrium consumption allocation, we must work with the effective Euler equation of the household problem. As we showed in section 3, this effective Euler equation is (19) to first order in the ε_t. Thus

$$\ln c_t = \ln(D_1 R) + \frac{\sum_{s=t}^{T} D_1^s \Delta\varepsilon_{s-t}}{\sum_{s'=t}^{T} D_1^{s'}} + \ln c_{t-1} + O(\varepsilon^2) \tag{40}$$

Iterating (40) from $\ln c_0$ to $\ln c_t$, we get

$$\ln c_t = t \ln D_1 R + \ln c_0 + \sum_{i=1}^{t} \frac{\sum_{s=i}^{T} D_1^s \Delta\varepsilon_{s-i}}{\sum_{s'=i}^{T} D_1^{s'}} + O(\varepsilon^2) \tag{41}$$

With a change of variables in the sum over the differences of the future weighting discount function and noting that $\Delta\varepsilon_0 = 0$, we can rewrite this as

$$\ln c_t = \ln c_t^0 + \sum_{i=1}^{t} \frac{\sum_{l=1}^{T-i} D_1^{i+l} \Delta\varepsilon_l}{\sum_{k=i}^{T} D_1^k} + O(\varepsilon^2). \tag{42}$$

Thus the realized utility at time t from the equilibrium consumption path is

$$U_t^* = \sum_{s=t}^{T} D_{s-t} \ln c_s = \sum_{s=t}^{T} D_{s-t} \ln c_s^0 + \sum_{s=t}^{T} \sum_{i=1}^{s} \sum_{l=1}^{T-i} \frac{D_1^{s-t+i+l} \Delta\varepsilon_l}{\sum_{k=i}^{T} D_1^k} + O(\varepsilon^2). \tag{43}$$

Notice that (39) and (43) are the same to zeroth order in the ε_t, so ΔU_t vanishes to zeroth order. Thus we can focus on the first-order terms of (43), which we will call

$$V_t = \sum_{s=t}^{T} \sum_{i=1}^{s} \sum_{l=1}^{T-i} \frac{D_1^{s-t+i+l} \Delta\varepsilon_l}{\sum_{k=i}^{T} D_1^k}. \tag{44}$$

Since V_t is a linear combination of the incremental changes $\Delta\varepsilon_s$ of the future weighting discount function, it is helpful to isolate the effect of each individual change. We define coefficients J_i^t such that

$$V_t = \sum_{i=1}^{T-1} J_i^t \Delta\varepsilon_i. \tag{45}$$

That is to say,

$$J_i^t = \left.\frac{\partial U_t^*}{\partial \Delta\varepsilon_i}\right|_{\varepsilon=0} \tag{46}$$

for $i = 1, \ldots, T-1$, noting that $\Delta\varepsilon_0 = 0$. Thus J_i^t measures how much $\Delta\varepsilon_i$ contributes to the equilibrium utility U_t^* at time t.

In appendix A we derive a convenient expression for the J_i^t:

$$J_i^t = \sum_{s=0}^{T-t} D_1^{i+s} \sum_{j=1}^{\min\{s+t,T-i\}} \frac{1}{\sum_{k=0}^{T-j} D_1^k} > 0, \tag{47}$$

where we assume that $D_1 > 0$. Thus an increase in $\Delta\varepsilon_i$ will unambiguously increase U_t^* at $\varepsilon = 0$. Note that if $i = T-1$, since $t \geq 1$, the inner sum in (47) reduces to a single term with $j = 1$, so

$$J_{T-1}^t = D_1^{T-1} \frac{\sum_{s=0}^{T-t} D_1^s}{\sum_{k=0}^{T-1} D_1^k} \leq D_1^{T-1} \leq D_1^{T-t}. \tag{48}$$

Note that the first inequality is strict for $t > 1$.

Another useful property of the J_i^t, shown in appendix B, is that, if $D_1 \in (0,1)$ they are strictly decreasing in i. For $t = 1, \ldots, T$ and $i = 1, \ldots, T-2$,

$$J_i^t > J_{i+1}^t. \tag{49}$$

Intuitively, it would make sense that the contribution of an incremental change $\Delta\varepsilon_i$ to equilibrium utility should get smaller the farther into the future the change in delays from i to $i+1$ gets.

If we express V_t in terms of the ε_i, we can write $\Delta U_t = U_t^* - U_t^c$ as

$$\Delta U_t = J_{T-1}^t \varepsilon_T + \sum_{i=2}^{T-1} (J_{i-1}^t - J_i^t)\varepsilon_i - \sum_{s=t}^{T} D_1^{s-t}\varepsilon_s + O(\varepsilon^2). \tag{50}$$

This shows that, to first order, the difference between the realized utility and the commitment utility at each t is a linear combination of the ε_i for $i = 2, \ldots, T$.

4.1 Comparing the commitment path with the equilibrium path at $t = 1$

The reason why we call the property that we are deriving conditions for in this section "almost Pareto dominance" by the commitment path is an interesting quirk of this discrete-time model. To first-order in the future weighting factors, utility at $t = 1$ is always the same along the equilibrium and commitment paths. This is a consequence of the fact that at $t = 0$ lifetime utility along the commitment path must, by definition, dominate lifetime utility along any other path, including the equilibrium path, so

$$\Delta U_0 \leq 0. \tag{51}$$

It follows from this that we must have

$$\Delta U_0 = O(\varepsilon^2) \tag{52}$$

since if

$$\Delta U_0 = \sum_{i=2}^{T} \Delta v_i^0 \varepsilon_i + O(\varepsilon^2)$$

for some $v_2^0, \ldots, v_T^0$, not all zero, then there would have to be some choice of the ε_i such that $\Delta U_0 > 0$. Let us define $c_{t,i}^1$ and $c_{t|0,i}^1$ for $i = 2, \ldots, T$ such that

$$c_t = c_t^0 + \sum_{i=2}^{T} c_{t,i}^1 \varepsilon_i + O(\epsilon^2) \tag{53}$$

and

$$c_{t|0} = c_t^0 + \sum_{i=2}^{T} c_{t|0,i}^1 \varepsilon_i + O(\epsilon^2). \tag{54}$$

But since $c_0 = c_{0|0}$,

$$\begin{aligned}
\Delta U_0 &= \ln c_0 + \sum_{t=1}^{T} D_t \ln c_t - \ln c_{0|0} - \sum_{t=1}^{T} D_t \ln c_{t|0} \\
&= \sum_{t=1}^{T} D_1^t (1+\varepsilon_t) \ln\left(\frac{c_t^0 + \sum_{i=2}^{T} c_{t,i}^1 \varepsilon_i}{c_t^0 + \sum_{j=2}^{T} c_{t|0,j}^1 \varepsilon_i} \right) + O(\varepsilon_t^2) \\
&= \sum_{t=1}^{T} D_1^t (1+\varepsilon_t) \sum_{i=2}^{T} \frac{c_{t,i}^1 - c_{t|0,i}^1}{c_t^0} \varepsilon_i + O(\varepsilon_t^2) \\
&= \sum_{t=1}^{T} D_1^t \sum_{i=2}^{T} \frac{c_{t,i}^1 - c_{t|0,i}^1}{c_t^0} \varepsilon_i + O(\varepsilon_t^2).
\end{aligned}$$

Likewise,

$$\begin{aligned}
\Delta U_1 &= \sum_{t=1}^{T} D_{t-1} \ln c_t - \sum_{t=1}^{T} D_{t-1} \ln c_{t|0} \\
&= \sum_{t=1}^{T} D_1^{t-1} \sum_{i=2}^{T} \frac{c_{t,i}^1 - c_{t|0,i}^1}{c_t^0} \varepsilon_i + O(\varepsilon_t^2).
\end{aligned}$$

Thus,

$$\Delta U_1 = \frac{1}{D_1} \Delta U_0 + O(\varepsilon^2). \tag{55}$$

Proposition 3. *If we define* $\Delta U_1 = U_1^* - U_1^c$ *then*

$$\Delta U_1 = O(\varepsilon^2).$$

This means U_1^c*, the utility on the commitment path at period 1, equals* U_1^**, the utility on the equilibrium path at period 1, to first order in* ε*.*

The preceding argument does not extend to the second-order terms of ΔU_0 and ΔU_1. Calculating the difference ΔU_1 to second order is beyond the scope of this paper. Caliendo and Findley (2019) have shown for $T = 2$ that ΔU_1 is always positive, which we now understand is a consequence of the second-order term always being positive. Thus it is never possible to have the commitment path dominate the equilibrium path for all t when $T = 2$. For large enough T, however, the second-order term can become negative. The conditions that we establish below for almost Pareto dominance would yield complete Pareto dominance at such large T by the commitment path.

4.2 Comparing the commitment path with the equilibrium path at $t > 1$

To sign the ΔU_t for $t = 2, \ldots, T$, it is helpful to isolate the contribution of the individual future weighting factors, so we define

$$\Delta U_t = \sum_{i=2}^{T} B_i^t \varepsilon_i + O(\varepsilon^2), \tag{56}$$

where the B_i^t represent the rate at which ΔU_t changes with ε_i when all the $\varepsilon_i = 0$:

$$B_i^t = \left. \frac{\partial \Delta U_t}{\partial \varepsilon_i} \right|_{\varepsilon=0}.$$

In other words, the B matrix is the Jacobian of the ΔU_t with respect to the future weights of the discount function. While this may not be immediately apparent, as the following proposition establishes, the signs of the B_i^t are unambiguous.

Proposition 4. *For $t = 2, \ldots, T$, if $D_1 \in (0, 1)$,*

a.

$$B_T^t = J_{T-1}^t - D_1^{T-t} < 0 \tag{57}$$

b. For $t \leq i < T$,

$$B_i^t = J_{i-1}^t - J_i^t - D_1^{i-t} < 0, \tag{58}$$

c. For $2 \leq i < t$,

$$B_i^t = J_{i-1}^t - J_i^t > 0. \tag{59}$$

The proofs of (57) and (59) follow immediately from the properties of the J_i^t detailed above. The proof of (58) is shown in appendix D.

If we write the B_i^t with t indexing the rows of the B matrix and i indexing its columns, since i and t both run from 2 to T, this will be a square matrix. To summarize Proposition 4, the matrix elements along and above the main diagonal will all be negative while the matrix elements below the main diagonal will be positive. This is true irrespective of the difference in definitions between the matrix elements of type a and type c. If we increase ε_i for $i \geq t$, then ΔU_t will decrease. On the other hand, if $i < t$, ΔU_t will increase.

We can understand this result as follows. From (46), $J^t_{i-1} - J^t_i$ is the contribution of ε_i to the equilibrium utility U^*_t for $i = 2, \ldots, T-1$, which we showed in Appendix B is positive. Likewise, J^t_{T-1} is the contribution of ε_T to U^*_t (since there is no ε_{T+1}), and this is also positive. Meanwhile, (34) shows that the contribution of a positive ε_i to the commitment utility U^c_t is D^{i-t}_1 for $i \geq t$, which is also positive, and zero otherwise. This last point is the key to the intuition behind Proposition 4. On the commitment path, the only effect of $\varepsilon_i > 0$ is to increase the consumption allocated to time i, so ε_i only contributes to the commitment utility at t if $i \geq t$. In contrast, on the equilibrium path, ε_i contributes to the equilibrium utility at all t for $t = 2, \ldots, T$, regardless of whether i comes before or after t.

The thrust of Proposition 4 is that, whenever ε_i has a nonzero contribution to the commitment utility at t, the contribution of ε_i to the equilibrium utility at t will always be of the same sign. However, if the first-order contribution of ε_i to the commitment utility is nonzero, it will always dominate the contribution of ε_i to the equilibrium utility.

While ε_i only impacts U^c_t through its effect on $\ln c_{i|0}$ and only if $i \geq t$, the effect of ε_i on the equilibrium utility at t is much more complicated since the effect of ε_i is spread over all the period utilities from $\ln c_2$ to $\ln c_T$. What makes this all the more remarkable is that, for example, ε_T only affects the household's decision-making for $t > 0$ through the choice of k_1 at $t = 0$. Notice that at $t = 0$, both the commitment and equilibrium paths start out the same. The only irreversible decision we actually make at $t = 0$ is the decision of how much of our $t = 0$ wealth to allocate to c_0 and how much to divide between $c_1, ..., c_T$. The ε_t will determine how we do this allocation over $c_1, ..., c_T$. Under commitment, we are committing to an allocation where each c_t is strictly a function of ε_t. But when we get to $t = 1$, we do not have to follow the plan that we had at $t = 0$, and in equilibrium we will not if the ε_t are nonzero. Instead, we decide again how we will allocate our wealth at $t = 1$ between c_1 and the $c_2, ..., c_T$.

And likewise, when we get to $t = 2$ we make a new plan for how much to consume at $t = 2$. Thus ε_{T-1} only affects the household's decision-making for $t > 1$ through the choice of k_2 at $t = 1$, and so on. The future weighting factor with the longest delay that appears in (19) at time t is ε_{T-t}. At later times, only weighting factors for shorter delays continue to appear in the Euler equation, so ε_i falls out of the Euler equations for $t > T - i$. Thereafter, a higher ε_i only impacts future consumptions through the choice to consume less at $T - i$.

This leaves a bigger pie remaining for the household to allocate amongst its selves at times later than $T-i$.

We can see how this works in relation to figure 1. For example in figure 1b, where $T = 10$ and there is a positive ε_8, there is a big spike in consumption at $t = 8$ on the commitment path. Thus the commitment utility will be higher for the selves that see this spike at $t \leq 8$. On the equilibrium path, consumption is slightly higher for all $t > T-8 = 2$, so the equilibrium utility is higher for all t. However, the effect of ε_8 on the equilibrium consumptions and the equilibrium utilities is small relative to the effect of ε_8 on $c_{8|0}$ and the commitment utility for $t \leq 8$. In equilibrium, after $t = 2$, ε_8 drops out of the calculation and provides no further reason for the household to save extra. Therefore it starts to smooth consumption and spreads the extra saving from $t = 2$ over the remaining 8 periods of life. This behavior is also observable in 1a where ε_2 is positive. In that case, starting from $t = 9$, ε_2 drops out and the household spreads the saving among the consumptions over the remaining periods of life. However, we see that since the saving is effectively divided between two periods, as opposed to seven periods in 1b, the increase in the period consumption level is larger. Note that to first order the superposition principle applies, so the effects of the discounting function in total will be the sum of the effects for each of the individual ε_i.

The sign of the matrix elements of type a in Proposition 4 are of most importance for understanding when the commitment path will almost Pareto dominate the equilibrium path or vice versa, so let us focus on why the B_T^t are always negative for $t = 2, \ldots, T$. For the equilibrium path, ε_T only appears in the initial Euler equation (19) that determines $\frac{c_1}{c_0}$, in which $\Delta\varepsilon_{T-1} = \varepsilon_T - \varepsilon_{T-1}$ is discounted by a factor of D_1^{T-1}. The equilibrium utility at t depends on $\ln c_s$ for $s = t, \ldots, T$, which all include $\ln \frac{c_1}{c_0}$. There are $T-t+1$ such terms, and they are also multiplied by the unperturbed marginal propensity to consume (MPC). After factoring out the unperturbed discount factor D_1^{T-1} mentioned previously, the denominator of the MPC is a sum of $T-1$ terms of comparable magnitude to the $T-t+1$ terms they are dividing, which yields a fraction less than one. Thus the magnitude of $\frac{\partial U_t^*}{\partial \varepsilon_T}$ is determined primarily by the discount factor D_1^{T-1}. Since this is smaller than $\frac{\partial U_t^c}{\partial \varepsilon_T} = D_1^{T-t}$, ε_T contributes to the equilibrium utility at t less than it contributes to the commitment utility at t, and $B_{T-1}^t < 0$.

Thus ε_T is of special significance of all the future weighting factors. An increase in ε_T will generate a big spike in consumption at the end of life along the commitment utility that will add to the commitment utility of all the household's selves. However, along the actual equilibrium path, this increase in ε_T will have a more muted effect. The initial self will reduce its consumption to enable the spike it is anticipating at the end of life, but thereafter all of the selves will take a bite of this extra saving. The concentrated dose of consumption in one period would have a bigger impact than spreading the consumption over all of the future selves. Holding the other future discount weights constant, if ε_T is pushed sufficiently high, all of the ΔU_t can be made negative for $t > 1$. Conversely, if ε_T is made sufficiently negative, all of the ΔU_t can be made positive for $t > 1$.[6]

Consequently, we can express the condition for ΔU_t to be negative (positive) in terms of a lower (upper) bound on ε_T. In formal terms, we can rearrange (56) such that $\Delta U_t < 0$ to first order in the future weighting factors iff

$$\varepsilon_T > -\sum_{i=2}^{T-1} \frac{B_i^t \varepsilon_i}{B_T^t}, \tag{60}$$

where the direction of the inequality follows from Proposition 4. Let us define the Pareto coefficients

$$P_i^t = -\frac{B_i^t}{B_T^t} \tag{61}$$

for $t = 2, \ldots, T$ and $i = 2, \ldots, T-1$.

Proposition 5. *To first order in ε, a necessary and sufficient condition for the commitment path to Pareto dominate the equilibrium path for all selves except $t = 1$ is that*

$$\varepsilon_T > \sum_{i=2}^{T-1} P_i^t \varepsilon_i. \tag{62}$$

Conversely, a necessary and sufficient condition for the equilibrium path to Pareto dominate the commitment path for all selves except $t = 0, 1$ is that

$$\varepsilon_T < \sum_{i=2}^{T-1} P_i^t \varepsilon_i. \tag{63}$$

Since P_i^t will have the same sign as B_i^t, it also follows from Proposition 4 that $P_i^t < 0$ iff $t \leq i < T$, and $P_i^t > 0$ iff $2 \leq i < t$ for $t = 2, \ldots, T$. If the future weights are heavy, it is trivial that $\Delta U_2 < 0$ since the P_i^2 are all negative. Likewise, if the future weights are light,

[6] Note that in continuous time we cannot vary the terminal future discount weight independently of the other weights while maintaining assumptions about the smoothness of the weighting function. This is one of the main advantages of working in discrete time.

$\Delta U_2 > 0$. On the other hand, the P_i^T will all be positive, so the threshold value of ε_T such that $\Delta U_T < 0$ will be strictly positive if the weights are heavy and strictly negative if the weights are light. For t in between 2 and T, the signs of the P_i^t will be mixed.

We should emphasize that, while the conditions for the ΔU_t to be positive or negative only specify a threshold value of ε_T, this does not imply that the other future weighting factors have no impact on the conditions for almost Pareto dominance. The threshold values of ε_T, i.e.

$$E_T^t = \sum_{i=2}^{T-1} P_i^t \varepsilon_i \tag{64}$$

for $t = 2, \ldots, T$, are themselves linear functions of the other future weighting factors. While we need to make additional assumptions to guarantee that E_T^T is the tightest threshold, i.e. either the most positive or the most negative, we can see that this threshold must increase (decrease) with all the other future weighting factors if the weights are heavy (light).

To sum up, the conditions we derived for ΔU_t to be negative for $t = 2, \ldots, T$ together combine to establish a sufficient condition for the commitment path to almost Pareto dominate the equilibrium path. Here we use the term *almost* Pareto dominate to emphasise that the sign of ΔU_1 is not determined to the first order of ε_t.

Example 6. *In this example we verify our results for a four-period model with T=3. As the first step, we set up the utility of the commitment plan U_t^c, the equilibrium plan U_t^*, and the difference between these two utilities ΔU_t. Then we establish the sufficient condition on ε_3 for the commitment path to almost Pareto dominate the equilibrium path and compare this to the conditions for a concave (convex) log consumption profile.*

In this four-period model,

$$U_t^* = \sum_{s=t}^{3} D_{s-t} \ln(c_s)$$

is the realized utility as of time t in equilibrium and

$$U_t^c = \sum_{s=t}^{3} D_{s-t} \ln(c_{s|0})$$

is the utility as of time t if the household commits to its original path. Finally,

$$\Delta U_t = U_t^* - U_t^c = \sum_{s=t}^{3} D_{s-t} \ln\left(\frac{c_s}{c_{s|0}}\right)$$

is the difference between these utilities.

Using (38) and (42) we have

$$
\begin{aligned}
\ln\left(\frac{c_1}{c_{1|0}}\right) &= \frac{D_1(1-D_1)\varepsilon_2 + D_1^2\varepsilon_3}{1+D_1+D_1^2} + O(\varepsilon^2) \\
\ln\left(\frac{c_2}{c_{2|0}}\right) &= \frac{D_1(1-D_1)\varepsilon_2 + D_1^2\varepsilon_3}{1+D_1+D_1^2} + \frac{D_1}{1+D_1}\varepsilon_2 - \varepsilon_2 + O(\varepsilon^2) \\
\ln\left(\frac{c_3}{c_{3|0}}\right) &= \frac{D_1(1-D_1)\varepsilon_2 + D_1^2\varepsilon_3}{1+D_1+D_1^2} + \frac{D_1}{1+D_1}\varepsilon_2 - \varepsilon_3 + O(\varepsilon^2)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\Delta U_1 &= \frac{D_1(1-D_1)\varepsilon_2 + D_1^2\varepsilon_3}{1+D_1+D_1^2} + D_1\left[\frac{D_1(1-D_1)\varepsilon_2 + D_1^2\varepsilon_3}{1+D_1+D_1^2} + \frac{D_1}{1+D_1}\varepsilon_2 - \varepsilon_2\right] \\
&\quad + D_1^2\left[\frac{D_1(1-D_1)\varepsilon_2 + D_1^2\varepsilon_3}{1+D_1+D_1^2} + \frac{D_1}{1+D_1}\varepsilon_2 - \varepsilon_3\right] \\
&= D_1(1-D_1)\varepsilon_2 + D_1^2\varepsilon_3 + D_1^2\varepsilon_2 - D_1\varepsilon_2 - D_1^2\varepsilon_3 \\
&= O(\varepsilon^2).
\end{aligned}
$$

Since ΔU_1 *vanishes to first order,*

$$
\begin{aligned}
\Delta U_2 &= -\frac{1}{D_1}\ln\left(\frac{c_1}{c_{1|0}}\right) \\
&= -\frac{1}{1+D_1+D_1^2}\left[(1-D_1)\varepsilon_2 + D_1\varepsilon_3\right] + O(\varepsilon^2) \\
\Delta U_3 &= D_1\frac{1-D_1^2+1+D_1+D_1^2}{(1+D_1+D_1^2)(1+D_1)}\varepsilon_2 - \frac{1+D_1}{1+D_1+D_1^2}\varepsilon_3 \\
&= \frac{1+D_1}{1+D_1+D_1^2}\left[\frac{2D_1+D_1^2}{(1+D_1)^2}\varepsilon_2 - \varepsilon_3\right] + O(\varepsilon^2)
\end{aligned}
$$

We will have

$$\Delta U_2 < 0$$

to first order of ε *if and only if*

$$(1-D_1)\varepsilon_2 + D_1\varepsilon_3 > 0$$

$$\varepsilon_3 > -\frac{1-D_1}{D_1}\varepsilon_2. \tag{65}$$

Likewise, we will have $\Delta U_3 < 0$ *to first order* ε *if and only if*

$$\varepsilon_3 > \left[1 - \frac{1}{(1+D_1)^2}\right]\varepsilon_2 \tag{66}$$

Thus to have $\Delta U_2 < 0$ *and* $\Delta U_3 < 0$*, we need*

$$\varepsilon_3 > \max\left\{\left[1 - \frac{1}{(1+D_1)^2}\right]\varepsilon_2, -\frac{1-D_1}{D_1}\varepsilon_2\right\} > 0 \tag{67}$$

since either both lower bounds are zero (if $\varepsilon_2 = 0$*) or one is positive and one is negative. Notice also that*

$$P_2^2 = 1 - \frac{1}{(1+D_1)^2} > 0,$$

and

$$P_2^3 = -\frac{1-D_1}{D_1} < 0,$$

consistent with Proposition 4 since P_i^t *has the same sign as* B_i^t.

For comparison, the concavity bounds on ε_2 *and* ε_3 *implied by (25) are* $\varepsilon_2 = \Delta\varepsilon_1 > 0$*, and*

$$\Delta\varepsilon_2 > \frac{D_1}{1+D_1}\Delta\varepsilon_1 = \frac{D_1}{1+D_1}\varepsilon_2.$$

Together, these strict concavity bounds on ε_2 *and* ε_3 *imply that*

$$\varepsilon_3 > \left[1 + \frac{1}{1+D_1}\right]\varepsilon_2 > \left[1 - \frac{1}{(1+D_1)^2}\right]\varepsilon_2 > -\frac{1-D_1}{D_1}\varepsilon_2.$$

Thus, when $T = 3$*, the necessary and sufficient conditions for a concave log consumption profile are also sufficient conditions for the commitment path to almost Pareto dominate the equilibrium path. Likewise, the conditions for a convex log consumption profile are sufficient conditions for the equilibrium path to almost Pareto dominate the commitment path.*

5 Discussion

In section 3, we developed the necessary condition for the consumption profile to be concave. And in section 4, we obtained the condition under which the commitment path almost Pareto dominates the equilibrium path. In Example 6, we also saw that the concavity condition for the lifecycle profile of log consumption implies the almost Pareto dominancy of the commitment path over the equilibrium path in a four-period version of the model.

To explore this, we will next show this last result also holds true in a five-period model. Our conjecture is that this result can be expanded to models with a longer horizon ($T > 4$). However, as the complexity of the following proof shows, if the conjecture is true, a general proof is beyond the scope of the present paper. However, in the context of a continuous time model, it is more straightforward to obtain an exact proof of the relation between a sufficient condition for concavity of log-consumption profile and a necessary condition for Pareto dominancy of the commitment path over the equilibrium path as is shown in Feigenbaum and Raei (2020).

A five-period model, If $T = 4$, based on our calculations in section 4, we can obtain the following Pareto bounds for period 2, period 3 and period 4.
The $t = 2$ Pareto bound is

$$\varepsilon_4 > -\frac{1 - D_1}{D_1^2}(\varepsilon_2 + D_1\varepsilon_3). \tag{68}$$

The $t = 3$ Pareto bound is

$$\varepsilon_4 > \frac{3D_1 + D_1^2 + 2D_1^3}{D_1 + D_1^2 + D_1^3}\varepsilon_2 - \frac{1 + D_1}{D_1 + D_1^2 + D_1^3}\varepsilon_3. \tag{69}$$

And the $t = 4$ Pareto bound is

$$\varepsilon_4 > P_2^4\varepsilon_2 + P_3^4\varepsilon_3 = \frac{3D_1 + D_1^2 + 2D_1^3}{(1 + D_1 + D_1^2)^2}\varepsilon_2 + \frac{2D_1^2 + D_1^3 + D_1^4}{(1 + D_1 + D_1^2)^2}\varepsilon_3 \tag{70}$$

Meanwhile, the concavity bounds for $T = 4$ are

$$\varepsilon_4 > \frac{D_1 - D_1^2}{1 + D_1 + D_1^2}\varepsilon_2 + \frac{1 + D_1 + 2D_1^2}{1 + D_1 + D_1^2}\varepsilon_3, \tag{71}$$

$$\varepsilon_3 > \frac{1 + 2D_1}{1 + D_1}\varepsilon_2, \tag{72}$$

and $\varepsilon_2 > 0$.

Let us suppose these concavity bounds are satisfied. Then (71) can be rewritten

$$\begin{aligned}
\varepsilon_4 &> \frac{D_1 - D_1^2}{1 + D_1 + D_1^2}\varepsilon_2 + \left(\frac{1 + D_1 + 2D_1^2}{1 + D_1 + D_1^2} - P_3^4\right)\varepsilon_3 + P_3^4\varepsilon_3 \\
&> \frac{D_1 - D_1^2}{1 + D_1 + D_1^2}\varepsilon_2 + \left(\frac{1 + D_1 + 2D_1^2}{1 + D_1 + D_1^2} - P_3^4\right)\frac{1 + 2D_1}{1 + D_1}\varepsilon_2 + P_3^4\varepsilon_3,
\end{aligned}$$

where we use (72) to obtain the second inequality. As we show in appendix E,

$$\frac{D_1 - D_1^2}{1 + D_1 + D_1^2} + \left(\frac{1 + D_1 + 2D_1^2}{1 + D_1 + D_1^2} - P_3^4\right)\frac{1 + 2D_1}{1 + D_1} > P_2^4 \tag{73}$$

Thus if the concavity bounds are satisfied we have

$$\varepsilon_4 > P_2^4\varepsilon_2 + P_3^4\varepsilon_3,$$

so the $t = 4$ Pareto bound is also satisfied.

Likewise, we can write the Pareto bound at $t = 3$ as

$$\varepsilon_4 > P_2^3\varepsilon_2 + P_3^3\varepsilon_3 \tag{74}$$

Suppose that both the concavity bounds and the $t = 4$ Pareto bound (70) are satisfied. We can rewrite the latter as

$$\varepsilon_4 > P_2^4\varepsilon_2 + (P_3^4 - P_3^3)\varepsilon_3 + P_3^3\varepsilon_3$$

Combining this with the concavity bound for ε_3, we obtain

$$\varepsilon_4 > P_2^4\varepsilon_2 + (P_3^4 - P_3^3)\frac{1 + 2D_1}{1 + D_1}\varepsilon_2 + P_3^3\varepsilon_3. \tag{75}$$

We show in appendix F that

$$P_2^4 + (P_3^4 - P_3^3)\frac{1 + 2D_1}{1 + D_1} > P_2^3,$$

so (69) immediately follows.

Thus if the concavity bounds for $T = 4$ are satisfied, the Pareto bounds for $T = 4$ are satisfied.

6 Concluding remarks

Present and future bias are defined as a form of time-inconsistency in which individuals' behavior regarding trade-offs in consumption at the beginning and end of the same time interval vary between the near future and the far future. The common approach for modeling this bias is with a relative discounting function, i.e. a form of discounting function which is a function of the time to consumption from the decision-making present. As a consequence, the optimal plan changes as an individual advances through the life span. A functional form that is widely used in the literature as a proxy for non-exponential discounting functions is the quasihyberbolic functional form, which is used to discuss the shape of the consumption profile and the preferences of different selves.

In this paper we proposed a general representation of relative discounting functions that allows us to focus on how the discounting function deviates from an exponential discounting function that will not exhibit time-inconsistency. We term the perturbation away from the exponential case a *future weighting factor* ε_t. This specific format of the discounting function provides a simple way to depict a future bias by having all ε_t be negative and decreasing for $t > 1$, and a present bias by having all ε_t be positive and increasing for $t > 1$. We find that the former is a necessary condition to have a convex log consumption profile and the latter is a necessary condition to have a concave log consumption profile.

Also, using the proposed future weighting functional form, we find a condition on ε_t under which the consumption profile that is determined in the first period of life will Pareto dominate the consumption profiles that are chosen at each period, starting from period two. This result is especially useful because this Pareto dominance is often used to motivate how one performs welfare analysis in these models with time-inconsistent preferences, where choosing a reference consumption plan for the analysis is a point of controversy in the literature.

An interesting extension of this paper could be to develop an analysis based on second order of ε_t in the first period. In other words, to investigate a condition for having the time-zero plan to Pareto dominate the equilibrium plan at the first period of life. Another potential path is with respect to the relation between the condition for the concavity of the log-consumption and the condition for the Pareto dominance of the commitment path. Providing a simple proof that can be extended beyond a five-period model would be very helpful. We explore this in the context of a continuous-time model in Feigenbaum and Raei (2020).

References

Frank Caliendo and T. Scott Findley. Commitment and welfare. Journal of Economic Behavior and Organization, 2019.

Mathias Dewatripont, Isabelle Brocas, and Juan Carrillo. Commitment devices under self-control problems: an overview. Ulb institutional repository, ULB – Universite Libre de Bruxelles, 2004.

James Feigenbaum and Sepideh Raei. Discounting faster or slower than exponential: Implications for consumption and welfare. working paper, 2020.

David Laibson. Hyperbolic discounting and consumption. PhD diss. Massachusetts Institute of Technology, 1994.

David Laibson. Golden eggs and hyperbolic discounting. The Quarterly Journal of Economics, 112(2):443–478, 1997.

David Laibson. Life-cycle consumption and hyperbolic discount functions. European Economic Review, 42(3):861 – 871, 1998.

David I Laibson. Hyperbolic discount functions, undersaving, and savings policy. Working Paper 5635, National Bureau of Economic Research, June 1996.

David I. Laibson, Andrea Repetto, Jeremy Tobacman, Robert E. Hall, William G. Gale, and George A. Akerlof. Self-control and saving for retirement. Brookings Papers on Economic Activity, 1998(1):91–196, 1998.

Jesus Marin-Solano and Jorge Navas. Non-constant discounting in finite horizon: The free terminal time case. Journal of Economic Dynamics and Control, 33(3):666–675, 2009.

Ted O'Donoghue and Matthew Rabin. The economics of immediate gratification. Journal of Behavioral Decision Making, 13(2):233–250.

Ted O'Donoghue and Matthew Rabin. Doing it now or later. American Economic Review, 89(1):103–124, March 1999.

Ted O'Donoghue and Matthew Rabin. Choice and procrastination. The Quarterly Journal of Economics, 116(1):121–160, 2001.

Ted O'Donoghue and Matthew Rabin. Present bias: Lessons learned and to be learned. American Economic Review, 105(5):273–79, May 2015.

R. H. Strotz. Myopia and inconsistency in dynamic utility maximization. Review of Economic Studies, 23(3):165–180, 1955.

Appendices

A Different Expressions for J_i^t notation

we know that

$$\Delta U_t = \sum_{s=t}^{T}\sum_{i=0}^{s-1}\sum_{j=1}^{T-i-1}\frac{D_1^{j+s-t}\Delta\varepsilon_j}{\sum_{k=0}^{T-i-1}D_1^k} - \sum_{s=t}^{T} D_1^{s-t}\varepsilon_s + O(\varepsilon^2)$$

Let

$$V_t^T = \sum_{s=t}^{T}\sum_{i=0}^{s-1}\sum_{j=1}^{T-i-1}\frac{D_1^{j+s-t}\Delta\varepsilon_j}{\sum_{k=0}^{T-i-1}D_1^k}$$

$$V_t^T = \sum_{s=0}^{T-t}\sum_{i=0}^{s+t-1}\sum_{j=1}^{T-i-1}\frac{D_1^{j+s}\Delta\varepsilon_j}{\sum_{k=0}^{T-i-1}D_1^k}$$

If we switch the first two summations, the s summation will commute with the j summation. Note that $t \geq 1$.

Let $S = \{(s,i) : 0 \leq s \leq T-t \wedge 0 \leq i \leq s+t-1\}$, so i runs from 0 to $T-1$. Let $S' = \{(s,i) : 0 \leq i \leq T-1 \wedge \max\{i+1-t, 0\} \leq s \leq T-t\}$. Let $(s,i) \in S$. Then $0 \leq s \leq T-t \wedge 0 \leq i \leq s+t-1 \leq T-1\}$. And we have $i+1-t \leq s \leq T-t$. We also have $0 \leq s \leq T-t$, so $\max\{0, i+1-t\} \leq s \leq T-t$. Thus $(s,i) \in S'$. Let $(s,i) \in S'$. Then $0 \leq i \leq T-1 \wedge \max\{i+1-t, 0\} \leq s \leq T-t$. Then $0 \leq s \leq T-t$. We also have $0 \leq i$ and $i+1-t \leq s$ so $i \leq s+t-1$. Therefore $(s,i) \in S$. Thus

$$V_t^T = \sum_{i=0}^{T-1}\sum_{s=\max\{0,i+1-t\}}^{T-t}\sum_{j=1}^{T-i-1}\frac{D_1^{j+s}\Delta\varepsilon_j}{\sum_{k=0}^{T-i-1}D_1^k}$$

$$V_t^T = \sum_{i=0}^{T-1}\sum_{j=1}^{T-i-1} D_1^j\Delta\varepsilon_j\frac{\sum_{s=\max\{0,i+1-t\}}^{T-t}D_1^s}{\sum_{k=0}^{T-i-1}D_1^k}$$

When $i = T-1$, the inner sum vanishes, so

$$V_t^T = \sum_{i=0}^{T-2}\sum_{j=1}^{T-i-1} D_1^j\Delta\varepsilon_j\frac{\sum_{s=\max\{0,i+1-t\}}^{T-t}D_1^s}{\sum_{k=0}^{T-i-1}D_1^k}$$

Let $i' = i + 1$, so $i = i' - 1$

$$V_t^T = \sum_{i'=1}^{T-1} \sum_{j=1}^{T-i'} D_1^j \Delta\varepsilon_j \frac{\sum_{s=\max\{0,i\prime-t\}}^{T-t} D_1^s}{\sum_{k=0}^{T-i'} D_1^k}$$

$$V_t^T = \sum_{i=1}^{T-1} \sum_{j=1}^{T-i} D_1^j \Delta\varepsilon_j \frac{\sum_{s=\max\{0,i-t\}}^{T-t} D_1^s}{\sum_{k=0}^{T-i} D_1^k}$$

$$V_t^T = \sum_{j=1}^{T-1} \sum_{i=1}^{T-j} D_1^j \Delta\varepsilon_j \frac{\sum_{s=\max\{0,i-t\}}^{T-t} D_1^s}{\sum_{k=0}^{T-i} D_1^k}$$

Switching the roles of i and j, we get

$$V_t^T = \sum_{i=1}^{T-1} \sum_{j=1}^{T-i} D_1^i \Delta\varepsilon_i \frac{\sum_{s=\max\{0,j-t\}}^{T-t} D_1^s}{\sum_{k=0}^{T-j} D_1^k}$$

Let us define

$$J_i^t = \sum_{j=1}^{T-i} \frac{\sum_{s=\max\{0,j-t\}}^{T-t} D_1^{i+s}}{\sum_{k=0}^{T-j} D_1^k} \tag{76}$$

Then

$$V_t^T = \sum_{i=1}^{T-1} J_i^t \Delta\varepsilon_i \tag{77}$$

Thus

$$\begin{aligned} V_t^T &= \sum_{i=1}^{T-1} J_i^t (\varepsilon_{i+1} - \varepsilon_i) \\ &= \sum_{i=1}^{T-1} J_i^t \varepsilon_{i+1} - \sum_{i=1}^{T-1} J_i^t \varepsilon_i \\ &= \sum_{i=2}^{T} J_{i-1}^t \varepsilon_i - \sum_{i=2}^{T-1} J_i^t \varepsilon_i \\ &= J_{T-1}^t \varepsilon_T + \sum_{i=2}^{T-1} (J_{i-1}^t - J_i^t) \varepsilon_i \end{aligned}$$

$$\Delta U_t = V_t^T - \sum_{s=t}^{T} D_1^{s-t} \varepsilon_s + O(\varepsilon^2)$$

$$\Delta U_t = J^t_{T-1}\varepsilon_T + \sum_{i=2}^{T-1}(J^t_{i-1} - J^t_i)\varepsilon_i - \sum_{s=t}^{T} D_1^{s-t}\varepsilon_s + O(\varepsilon^2) \tag{78}$$

We can show that the way we define J^t_i notation in the paper is equivalent to the formula we have here by switching the indices in the following way. Let us start with the definition of j^t_i that we used above:

$$J^t_i = \sum_{j=1}^{T-i} \frac{\sum_{s=\max\{0,j-t\}}^{T-t} D_1^{i+s}}{\sum_{k=0}^{T-j} D_1^k}$$

Let $S = \{(j,s) : 1 \leq j \leq T-i \wedge \max\{0, j-t\} \leq s \leq T-t\}$. Let $S' = \{(j,s) : 0 \leq s \leq T-t \wedge 1 \leq j \leq \min\{s+t, T-i\}\}$. Let $(j,s) \in S$. Then $1 \leq j \leq T-i \wedge \max\{0, j-t\} \leq s \leq T-t$. Then we have $0 \leq s \leq T-t$. We also have $1 \leq j \leq T-i$. And we have $j-t \leq s$, so $1 \leq j \leq \min\{s+t, T-i\}$. Thus $(j,s) \in S'$.

Suppose $(j,s) \in S'$. Then we have $0 \leq s \leq T-t \wedge 1 \leq j \leq \min\{s+t, T-i\}$. Thus $1 \leq j \leq \min\{s+t, T-i\}$. We also have $0 \leq s$ and $j \leq s+t$, so $s \geq \max\{0, j-t\}$, and so $\max\{0, j-t\} \leq s \leq T-t$. Thus we can write

$$J^t_i = \sum_{s=0}^{T-t} D_1^{i+s} \sum_{j=1}^{\min\{s+t,T-i\}} \frac{1}{\sum_{k=0}^{T-j} D_1^k}. \tag{79}$$

B J^t_i is strictly decreasing in i

$$J^t_i = \sum_{s=0}^{T-t} D_1^{i+s} \sum_{j=1}^{\min\{s+t,T-i\}} \frac{1}{\sum_{k=0}^{T-j} D_1^k}$$

$$\begin{aligned} J^t_{i+1} &= \sum_{s=0}^{T-t} D_1^{i+1+s} \sum_{j=1}^{\min\{s+t,T-i-1\}} \frac{1}{\sum_{k=0}^{T-j} D_1^k} \\ &\leq \sum_{s=0}^{T-t} D_1^{i+1+s} \sum_{j=1}^{\min\{s+t,T-i\}} \frac{1}{\sum_{k=0}^{T-j} D_1^k} \\ &< \sum_{s=0}^{T-t} D_1^{i+s} \sum_{j=1}^{\min\{s+t,T-i\}} \frac{1}{\sum_{k=0}^{T-j} D_1^k} = J^t_i, \end{aligned}$$

where this last inequality assumes $D_1 \in (0,1)$.

C Direct Proof of Proposition 3

The proof amounts to showing that the first-order expansion of equilibrium utility at $t = 1$ is

$$V_1 = \sum_{s=2}^{T} D_1^{s-1} \varepsilon_s, \tag{80}$$

where the right-hand side is lifetime utility at $t = 1$ under commitment, also to first order.

In Appendix A, we show that another expression for J_i^t is

$$J_i^t = \sum_{j=1}^{T-i} \frac{\sum_{s=\max\{0,j-t\}}^{T-t} D_1^{i+s}}{\sum_{k=0}^{T-j} D_1^k}.$$

For $t = 1$, this simplifies to

$$J_i^1 = \sum_{j=1}^{T-i} \frac{\sum_{s=j-1}^{T-1} D_1^{i+s}}{\sum_{k=0}^{T-j} D_1^k}. \tag{81}$$

We can factor D_1^{i+j-1} out of the summation over s, leaving

$$\frac{\sum_{k'=0}^{T-j} D_1^{k'}}{\sum_{k=0}^{T-j} D_1^k} = 1.$$

Thus

$$J_i^1 = D_1^i \sum_{j=1}^{T-i} D_1^{j-1},$$

and from (45)

$$V_1^T = \sum_{i=1}^{T-1} \sum_{j=1}^{T-i} D_1^{i+j-1} \Delta\varepsilon_i.$$

Let $l = i + j$, so $j = l - i$. Then

$$V_1^T = \sum_{i=1}^{T-1} \sum_{l=i+1}^{T} D_1^{l-1} \Delta\varepsilon_i.$$

Finally, if we commute the sums,

$$V_1^T = \sum_{l=2}^{T} D_1^{l-1} \sum_{i=1}^{l-1} \Delta\varepsilon_i,$$

the result follows from the fact that

$$\varepsilon_l = \sum_{i=1}^{l-1} \Delta\varepsilon_i.$$

D Proof of Inequality (58)

For $t = 2, ..., T$ and $i = 1, ..., T-1$, let us define

$$M_i^t = J_{i-1}^t - J_i^t. \tag{82}$$

Using (47),

$$M_i^t = D_1^{i-1}\left[(1-D_1)\sum_{s=0}^{T-t} D_1^s \sum_{j=1}^{\min\{s+t,T-i\}} \frac{1}{\sum_{k=0}^{T-j} D_1^k} + \frac{\sum_{s=\max\{T-i-t+1,0\}}^{T-t} D_1^s}{\sum_{k=0}^{i-1} D_1^k}\right]. \tag{83}$$

An equivalent but more convenient expression is obtained by rearranging the sums in the first term:

$$M_i^t = D_1^{i-1}\left[(1-D_1)\sum_{j=1}^{T-i} \frac{\sum_{s=\max\{0,j-t\}}^{T-t} D_1^s}{\sum_{k=0}^{T-j} D_1^k} + \frac{\sum_{s=\max\{T-i-t+1,0\}}^{T-t} D_1^s}{\sum_{k=0}^{i-1} D_1^k}\right] \tag{84}$$

Let

$$S = \{(s,j) : 0 \leq s \leq T-t \wedge 1 \leq j \leq \min\{s+t, T-i\}\}$$

and

$$S' = \{(s,j) : 1 \leq j \leq T-i \wedge \max\{0, j-t\} \leq s \leq T-t\}.$$

Suppose that $(s,j) \in S$. Then $0 \leq s \leq T-t \wedge 1 \leq j \leq \min\{s+t, T-i\}$. Thus $1 \leq j \leq T-i$ and $0 \leq s \leq T-t$. Plus $j \leq s+t$, so $s \geq j-t$. Thus $\max\{0, j-t\} \leq s \leq T-t$. Therefore, $(s,j) \in S'$.

Suppose that $(s,j) \in S'$. Then $1 \leq j \leq T-i \wedge \max\{0, j-t\} \leq s \leq T-t\}$. Thus $0 \leq s \leq T-t$. And $1 \leq j \leq T-i$, and $j-t \leq s$, so $j \leq s+t$. Therefore, $1 \leq j \leq \min\{s+t, T-i\}$, so $(s,j) \in S$. Thus, we can rewrite (83) as (84).

If $D_1 \in (0,1)$ then we can write

$$M_s^t < D_1^{s-1}\left[(1-D_1)\sum_{j=1}^{T-s}\frac{\sum_{i=j-t}^{T-t}D_1^i}{\sum_{k=0}^{T-j}D_1^k} + \frac{\sum_{i=T-s-t+1}^{T-t}D_1^i}{\sum_{k=0}^{s-1}D_1^k}\right].$$

This inequality is strict because $t \geq 2$, so there will be at least one positive term with $s < 0$ in the first sum that is not included in the first sum of (84). Then

$$\begin{aligned}
M_s^t &< D_1^{s-1}\left[(1-D_1)\sum_{j=1}^{T-s}D_1^{j-t}\frac{\sum_{i=0}^{T-j}D_1^i}{\sum_{k=0}^{T-j}D_1^k} + D_1^{T-s-t+1}\frac{\sum_{i=0}^{i-1}D_1^i}{\sum_{k=0}^{s-1}D_1^k}\right] \\
&= D_1^{s-1}\left[(1-D_1)\sum_{j=1}^{T-s}D_1^{j-t} + D_1^{T-s-t+1}\right] \\
&= (1-D_1)D_1^{s-1}\sum_{j=1}^{T-s}D_1^{j-t} + D_1^{T-t}
\end{aligned}$$

We can use this result to determine the sign of B_s^t for $t \leq s < T$. Since $s \geq t$,

$$\begin{aligned}
B_s^t &= M_s^t - D_1^{s-t} \\
&< (1-D_1)D_1^{s-1}\sum_{j=1}^{T-s}D_1^{j-t} + D_1^{T-t} - D_1^{s-t} \\
&= (1-D_1)\left[D_1^{s-1}\sum_{j=1}^{T-s}D_1^{j-t} + D_1^{s-t}\frac{D_1^{T-t-s+t}-1}{1-D_1}\right] \\
&= (1-D_1)\left[D_1^{s-1}\sum_{j=1}^{T-s}D_1^{j-t} + D_1^{s-t}\frac{D_1^{T-s}-1}{1-D_1}\right] \\
&= (1-D_1)\sum_{j=1}^{T-s}\left[D_1^{s-1}D_1^{j-t} - D_1^{s-t}D_1^{j-1}\right] = 0
\end{aligned}$$

E Proof of inequality (73)

we defined P_2^4 as

$$P_2^4 = \frac{3D_1 + D_1^2 + 2D_1^3}{(1+D_1+D_1^2)^2} \tag{85}$$

therefore

$$\begin{aligned}
P_2^4 - \frac{D_1 - D_1^2}{1 + D_1 + D_1^2} &= \frac{3D_1 + D_1^2 + 2D_1^3 - D_1(1 - D_1)(1 + D_1 + D_1^2)}{(1 + D_1 + D_1^2)^2} \\
&= \frac{3D_1 + D_1^2 + 2D_1^3 - D_1(1 - D_1^3)}{(1 + D_1 + D_1^2)^2} \\
&= \frac{3D_1 + D_1^2 + 2D_1^3 - D_1 + D_1^4}{(1 + D_1 + D_1^2)^2} \\
&= \frac{2D_1 + D_1^2 + 2D_1^3 + D_1^4}{(1 + D_1 + D_1^2)^2} \geq 0
\end{aligned}$$

also we have

$$\begin{aligned}
(1 + D_1 + D_1^2)^2 &= (1 + D_1)^2 + 2D_1^2(1 + D_1) + D_1^4 \\
&= 1 + 2D_1 + D_1^2 + 2D_1^2 + 2D_1^3 + D_1^4 \\
&= 1 + 2D_1 + 3D_1^2 + 2D_1^3 + D_1^4
\end{aligned}$$

hence

$$P_2^4 - \frac{D_1 - D_1^2}{1 + D_1 + D_1^2} = \frac{2D_1 + D_1^2 + 2D_1^3 + D_1^4}{(1 + D_1 + D_1^2)^2} < \frac{1 + 2D_1 + 3D_1^2 + 2D_1^3 + D_1^4}{(1 + D_1 + D_1^2)^2} = 1 \qquad (86)$$

with equality only if $D_1 = 0$.

Also, we defined P_3^4 as

$$P_3^4 = \frac{2D_1^2 + D_1^3 + D_1^4}{(1 + D_1 + D_1^2)^2}$$

therefore we can have the following

$$\begin{aligned}
\left(\frac{1+D_1+2D_1^2}{1+D_1+D_1^2}-P_3^4\right)\frac{1+2D_1}{1+D_1} &= \frac{(1+D_1)(1+D_1+D_1^2+D_1^3)}{(1+D_1+D_1^2)^2}\frac{1+2D_1}{1+D_1}\\
&= \frac{(1+2D_1)(1+D_1+D_1^2+D_1^3)}{(1+D_1+D_1^2)^2}\\
&= \frac{1+D_1+D_1^2+D_1^3+2D_1+2D_1^2+2D_1^3+2D_1^4}{(1+D_1+D_1^2)^2}\\
&= \frac{1+3D_1+3D_1^2+3D_1^3+2D_1^4}{(1+D_1+D_1^2)^2}\\
&= 1+\frac{D_1+D_1^3+D_1^4}{(1+D_1+D_1^2)^2}\\
&> 1 > P_2^4-\frac{D_1-D_1^2}{1+D_1+D_1^2}
\end{aligned}$$

in which we used (86) to drive the last inequality.

F Proof of inequality (75)

Here we show that $h(D_1)$ which is defined as

$$h(D_1)=(P_3^4-P_3^3)\frac{1+2D_1}{1+D_1}+P_2^4$$

satisfies

$$h(D_1)-P_2^3>0$$

As a reminder,

$$P_2^3=\frac{3+D_1+2D_1^2}{1+D_1+D_1^2}>\frac{3D_1+D_1^2+2D_1^3}{(1+D_1+D_1^2)^2}=P_2^4$$

and

$$P_3^3=-\frac{1+D_1}{D_1+D_1^2+D_1^3}<0<\frac{2D_1^2+D_1^3+D_1^4}{(1+D_1+D_1^2)^2}=P_3^4$$

$$
\begin{aligned}
P_3^4 - P_3^3 &= \frac{2D_1^2 + D_1^3 + D_1^4}{(1 + D_1 + D_1^2)^2} + \frac{1 + D_1}{D_1 + D_1^2 + D_1^3} \\
&= \frac{2D_1^3 + D_1^4 + D_1^5 + (1 + D_1)(1 + D_1 + D_1^2)}{D_1(1 + D_1 + D_1^2)^2} \\
&= \frac{2D_1^3 + D_1^4 + D_1^5 + 1 + 2D_1 + 2D_1^2 + D_1^3}{D_1(1 + D_1 + D_1^2)^2} \\
&= \frac{1 + 2D_1 + 2D_1^2 + 3D_1^3 + D_1^4 + D_1^5}{D_1(1 + D_1 + D_1^2)^2}
\end{aligned}
$$

and

$$
\begin{aligned}
P_2^4 - P_2^3 &= \frac{3D_1 + D_1^2 + 2D_1^3}{(1 + D_1 + D_1^2)^2} - \frac{3 + D_1 + 2D_1^2}{1 + D_1 + D_1^2} \\
&= \frac{3 + D_1 + 2D_1^2}{1 + D_1 + D_1^2}\left[\frac{D_1}{1 + D_1 + D_1^2} - 1\right] \\
&= -\frac{(3 + D_1 + 2D_1^2)(1 + D_1^2)}{(1 + D_1 + D_1^2)^2} \\
&= -\frac{3 + D_1 + 2D_1^2 + 3D_1^2 + D_1^3 + 2D_1^4}{(1 + D_1 + D_1^2)^2} \\
&= -\frac{3 + D_1 + 5D_1^2 + D_1^3 + 2D_1^4}{(1 + D_1 + D_1^2)^2}
\end{aligned}
$$

hence we have

$$
\begin{aligned}
h(D_1) - P_2^3 &= (P_3^4 - P_3^3)\frac{1 + 2D_1}{1 + D_1} + P_2^4 - P_2^3 \\
&= \frac{1 + 2D_1 + 2D_1^2 + 3D_1^3 + D_1^4 + D_1^5}{D_1(1 + D_1 + D_1^2)^2}\frac{1 + 2D_1}{1 + D_1} - \frac{3 + D_1 + 5D_1^2 + D_1^3 + 2D_1^4}{(1 + D_1 + D_1^2)^2} \\
&= \frac{(1 + 2D_1 + 2D_1^2 + 3D_1^3 + D_1^4 + D_1^5)(1 + 2D_1) - (3 + D_1 + 5D_1^2 + D_1^3 + 2D_1^4)D_1(1 + D_1)}{D_1(1 + D_1 + D_1^2)^2(1 + D_1)}
\end{aligned}
$$

The numerator is

$$
\begin{aligned}
& 1 + 2D_1 + 2D_1^2 + 3D_1^3 + D_1^4 + D_1^5 + 2D_1 + 4D_1^2 + 4D_1^3 + 6D_1^4 + 2D_1^5 + 2D_1^6 \\
& -3D_1 - D_1^2 - 5D_1^3 - D_1^4 - 2D_1^5 - 3D_1^2 - D_1^3 - 5D_1^4 - D_1^5 - 2D_1^6 \\
={}& 1 + D_1 + 2D_1^2 + D_1^3 + D_1^4 \\
={}& (1 + D_1 + D_1^2)(1 + D_1^2)
\end{aligned}
$$

therefore

$$
\begin{aligned}
h(D_1) - P_2^3 &= \frac{(1 + D_1 + D_1^2)(1 + D_1^2)}{D_1(1 + D_1 + D_1^2)^2(1 + D_1)} = \frac{1 + D_1^2}{D_1(1 + D_1)(1 + D_1 + D_1^2)} \\
&= \frac{1 + D_1^2}{D_1 + 2D_1^2 + 2D_1^3 + D_1^4} > 0
\end{aligned}
$$